

BRAIDED SWEEDLER COHOMOLOGY

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ABSTRACT. We introduced a braided Sweedler cohomology, which is adequate to work with the H -braided cleft extensions studied in [G-G1].

INTRODUCTION

In [Sw] a cohomology theory $H^*(H, A)$ for a commutative module algebra A over a cocommutative Hopf algebra H was introduced. This cohomology is related to those of groups and Lie algebras in the following sense. When H is a group algebra $k[G]$, then $H^*(H, A)$ is canonically isomorphic to the group cohomology of G in the multiplicative group of invertible elements of A , and when H is the enveloping algebra $U(L)$ of a Lie algebra L , then $H^n(H, A)$ is canonically isomorphic to the cohomology of L in the underlying vector space of A , for all $n \geq 2$.

One of the main properties of the Sweedler cohomology is that there is a bijective correspondence between $H^2(H, A)$ and the equivalence classes of H -cleft extensions of A . This result was extended in [D1], where it was shown that the hypothesis of commutativity of A can be removed.

Let H be a braided bialgebra. In [G-G1] a notion of clef extension of an H -braided module algebra (A, s) was presented (for the definitions see Section 1). This concept is more general than the one defined in [B-C-M] still when H is a standard Hopf algebra. Assume that H is a braided cocommutative Hopf algebra. In this paper we present a braided version of the Sweedler cohomology in order to classify the cleft extensions introduced in [G-G1].

The paper is organized as follows: Section 1 is devoted to review some notions from [G-G1] and to introduce some concepts that we will need later. In Section 2, we define, by means of an explicit complex, the braided Sweedler cohomology of a braided cocommutative Hopf algebra H with coefficients in an H -braided module algebra A . When H is a cocommutative standard Hopf algebra and H is an usual module algebra, our complex reduced to the classical one of Sweedler. In Section 3 we show that the second cohomology group of our complex classifies the cleft extensions of an H -braided module algebra (A, s) . In Section 4 we prove that when H is a group algebra $k[G]$, the braided Sweedler cohomology of H with coefficients in an H -braided module algebra (A, s) coincides with a variant of the group homology of G with coefficients in the multiplicative group of invertible elements of A and in Section 5 we prove a similar result for the cohomology groups of degree greater than 1, when H is the enveloping algebra of a Lie algebra L . In Section 6 we show that in order to compute the cohomology mentioned in the previous section, a Chevalley-Eilenberg type complex can be used. Finally, in Section 7, we calculate all the cleft extensions in a particular case.

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1. PRELIMINARIES

In this article we work in the category of vector spaces over a field k . Then we assume implicitly that all the maps are k -linear and all the algebras and coalgebras are over k . The tensor product over k is denoted by \otimes , without any subscript, and the category of k -vector spaces is denoted by $\mathcal{V}\text{ect}$. Given a vector space V and $n \geq 1$, we let V^n denote the n -fold tensor power $V \otimes \cdots \otimes V$. Given vector spaces U, V, W and a map $f: V \rightarrow W$ we write $U \otimes f$ for $\text{id}_U \otimes f$ and $f \otimes U$ for $f \otimes \text{id}_U$. We assume that the algebras are associative unitary and the coalgebras are coassociative counitary. Given an algebra A and a coalgebra C , we let $\mu: A \otimes A \rightarrow A$, $\eta: k \rightarrow A$, $\Delta: C \rightarrow C \otimes C$ and $\epsilon: C \rightarrow k$ denote the multiplication, the unit, the comultiplication and the counit, respectively, specified with a subscript if necessary.

Some of the results of this paper are valid in the context of monoidal categories. In fact we use the nowadays well known graphic calculus for monoidal and braided categories. As usual, morphisms will be composed from up to down and tensor products will be represented by horizontal concatenation in the corresponding order. The identity map of a vector space will be represented by a vertical line. Given an algebra A , the diagrams

$$\begin{array}{c} \text{Y} \\ \text{!} \end{array} \quad \text{and} \quad \begin{array}{c} \text{Y} \\ \text{!} \end{array}$$

stand for the multiplication map, the unit and the action of A on a left A -module, respectively, and for a coalgebra C , the comultiplication and the counit will be represented by the diagrams

$$\begin{array}{c} \text{Y} \\ \text{!} \end{array} \quad \text{and} \quad \begin{array}{c} \text{Y} \\ \text{!} \end{array},$$

respectively. The maps c and s , which appear at the beginning of Subsection 1.1, will be represented by the diagrams

$$\begin{array}{c} \text{X} \\ \text{X} \end{array} \quad \text{and} \quad \begin{array}{c} \text{X} \\ \text{X} \end{array},$$

respectively. Finally, any other map $g: V \rightarrow W$ will be geometrically represented by the diagram

$$\begin{array}{c} \text{Y} \\ \text{!} \end{array}.$$

Remark 1.1. A Sweedler cohomology for module algebras in a symmetric tensor category was presented in [A-F-G]. The version study by us is different of this one, because of the existence of a transposition involved in our definition of H -braided module algebras (see section 1).

Let V, W be vector spaces and let $c: V \otimes W \rightarrow W \otimes V$ be a map. Recall that:

- If V is an algebra, then c is compatible with the algebra structure of V if $c \circ (\eta \otimes W) = W \otimes \eta$ and $c \circ (\mu \otimes W) = (W \otimes \mu) \circ (c \otimes V) \circ (V \otimes c)$.
- If V is a coalgebra, then c is compatible with the coalgebra structure of V if $(W \otimes \epsilon) \circ c = \epsilon \otimes W$ and $(W \otimes \Delta) \circ c = (c \otimes V) \circ (V \otimes c) \circ (\Delta \otimes W)$.

Of course, there are similar compatibilities when W is an algebra or a coalgebra.

Next we recall briefly the concepts of braided bialgebra and braided Hopf algebra following the presentation given in [T1]. For a study of braided Hopf algebras we refer to [T1], [T2], [L1], [F-M-S], [A-S], [D2], [So] and [B-K-L-T].

Definition 1.2. A *braided bialgebra* is a vector space H endowed with an algebra structure, a coalgebra structure and a braiding operator $c \in \text{Aut}_k(H^2)$ (called the *braid* of H), such that c is compatible with the algebra and coalgebra structures of H , $\Delta \circ \mu = (\mu \otimes \mu) \circ (H \otimes c \otimes H) \circ (\Delta \otimes \Delta)$, η is a coalgebra morphism and ϵ is

an algebra morphism. Furthermore, if there exists a map $S: H \rightarrow H$, which is the inverse of the identity map for the convolution product, then we say that H is a *braided Hopf algebra* and we call S the *antipode* of H .

Usually H denotes a braided bialgebra, understanding the structure maps, and c denotes its braid. If necessary, we will use notations as c_H , μ_H , etcetera.

1.1. H -module algebras and H -module coalgebras. Let H be a braided bialgebra. Recall from [G-G1, Section 5] that a *left H -braided space* (V, s) is a vector space V , endowed with a bijective map $s: H \otimes V \rightarrow V \otimes H$, which is compatible with the bialgebra structure of H and satisfies

$$(s \otimes H) \circ (H \otimes s) \circ (c \otimes V) = (V \otimes c) \circ (s \otimes H) \circ (H \otimes s)$$

(compatibility of s with the braid). Let (V', s') be another left H -braided space. A k -linear map $f: V \rightarrow V'$ is said to be a *morphism of left H -braided spaces*, from (V, s) to (V', s') , if $(f \otimes H) \circ s = s' \circ (H \otimes f)$. We let \mathcal{LB}_H denote the category of all left H -braided spaces. It is easy to check that this is a monoidal category with:

- unit (k, τ) , where $\tau: H \otimes k \rightarrow k \otimes H$ is the flip,
- tensor product $(V, s_V) \otimes (U, s_U) := (V \otimes U, s_{V \otimes U})$, where $s_{V \otimes U}$ is the map $s_{V \otimes U} := (V \otimes s_U) \circ (s_V \otimes U)$,
- the usual associativity and unit constraints.

Let A be an algebra. We recall from [G-G1] that a *left transposition* is a bijective map $s: H \otimes A \rightarrow A \otimes H$, satisfying:

- (1) (A, s) is a left H -braided space,
- (2) s is compatible with the algebra structure of A .

Remark 1.3. It is easy to check that an algebra in \mathcal{LB}_H , also called a *left H -braided algebra*, is a pair (A, s) , consisting of an algebra A and a left transposition $s: H \otimes A \rightarrow A \otimes H$. Let (A', s') be another left H -braided algebra. A map $f: A \rightarrow A'$ is a *morphism of left H -braided algebras*, from (A, s) to (A', s') , if it is a morphism of standard algebras and $(f \otimes H) \circ s = s' \circ (H \otimes f)$.

Definition 1.4. Let C be a coalgebra. A *left transposition* of H on C is a bijective map $s: H \otimes C \rightarrow C \otimes H$, satisfying:

- (1) (C, s) is a left H -braided space,
- (2) s is compatible with the coalgebra structure of C .

Remark 1.5. It is easy to check that a coalgebra in \mathcal{LB}_H , also called a *left H -braided coalgebra*, is a pair (C, s) consisting of a coalgebra C and a left transposition $s: H \otimes C \rightarrow C \otimes H$. Let (C', s') be another left H -braided coalgebra. A map $f: C \rightarrow C'$ is a *morphism of left H -braided coalgebras*, from (C, s) to (C', s') , if it is a morphism of standard coalgebras and $(f \otimes H) \circ s = s' \circ (H \otimes f)$.

Note that (H, c) is an algebra in \mathcal{LB}_H . Hence, one can consider left and right (H, c) -modules in this monoidal category. To abbreviate we will say that (V, s) is a *left H -braided module* or simply a *left H -module* to mean that it is a left (H, c) -module in \mathcal{LB}_H . It is easy to check that a left H -braided space (V, s) is a left H -module if and only if V is a standard left H -module and $s \circ (H \otimes \rho) = (\rho \otimes H) \circ (H \otimes s) \circ (c \otimes V)$, where ρ denotes the action of H on V . Furthermore, a map $f: V \rightarrow V'$ is a *morphism of left H -modules*, from (V, s) to (V', s') , if it is H -linear and $(f \otimes H) \circ s = s' \circ (H \otimes f)$. We let ${}_H(\mathcal{LB}_H)$ denote the category of left H -braided modules.

Given left H -modules (V, s_V) and (U, s_U) , with actions ρ_V and ρ_U respectively, we let $\rho_{V \otimes U}$ denote the diagonal action

$$\rho_{V \otimes U} := (\rho_V \otimes \rho_U) \circ (H \otimes s_V \otimes U) \circ (\Delta_H \otimes V \otimes U).$$

In the following proposition we show in particular that (k, τ) is a left H -module via the trivial action and that $(V, s_V) \otimes (U, s_U)$ is a left H -module via $\rho_{V \otimes U}$.

Proposition 1.6 (G-G, Proposition 5.6). *The category ${}_H(\mathcal{LB}_H)$, of left H -braided modules, endowed with the usual associativity and unit constraints, is monoidal.*

Definition 1.7 (G-G, Definition 5.7). We say that (A, s) is a *left H -braided module algebra* or simply a *left H -module algebra* it is an algebra in ${}_H(\mathcal{LB}_H)$.

Remark 1.8. (A, s) is a left H -module algebra if and only if the following facts hold:

- (1) A is an algebra and a standard left H -module,
- (2) s is a left transposition of H on A ,
- (3) $s \circ (H \otimes \rho) = (\rho \otimes H) \circ (H \otimes s) \circ (c \otimes A)$,
- (4) $\mu_A \circ (\rho \otimes \rho) \circ (H \otimes s \otimes A) \circ (\Delta_H \otimes A^2) = \rho \circ (H \otimes \mu_A)$,
- (5) $h \cdot 1 = \epsilon(h)1$ for all $h \in H$,

where ρ denotes the action of H on A .

Let (A', s') be another left H -module algebra. A map $f: A \rightarrow A'$ is a *morphism of left H -module algebras*, from (A, s) to (A', s') , if it is an H -linear morphism of standard algebras that satisfies $(f \otimes H) \circ s = s' \circ (H \otimes f)$.

Definition 1.9. We say that (C, s) is a *left H -braided module coalgebra* or simply a *left H -module coalgebra* if it is a coalgebra in ${}_H(\mathcal{LB}_H)$.

Remark 1.10. (C, s) is a left H -module coalgebra if and only if the following facts hold:

- (1) C is a coalgebra and a standard left H -module,
- (2) s is a left transposition of H on C ,
- (3) $s \circ (H \otimes \rho) = (\rho \otimes H) \circ (H \otimes s) \circ (c \otimes C)$,
- (4) $(\rho \otimes \rho) \circ (H \otimes s \otimes C) \circ (\Delta_H \otimes \Delta_C) = \Delta_C \circ \rho$,
- (5) $\epsilon(h \cdot c) = \epsilon(h)\epsilon(c)$ for all $h \in H$ and $c \in C$,

where ρ denotes the action of H on C .

Let (C', s') be another left H -module coalgebra. A map $f: C \rightarrow C'$ is a *morphism of left H -module coalgebras*, from (C, s) to (C', s') , if it is an H -linear morphism of standard coalgebras that satisfies $(f \otimes H) \circ s = s' \circ (H \otimes f)$.

Let $H \otimes^s C$ be the coalgebra with underlying vector space $H \otimes C$, comultiplication map $\Delta_{H \otimes^s C} := (H \otimes s \otimes C) \circ (\Delta_H \otimes \Delta_C)$ and counit map $\epsilon_{H \otimes^s C} := \epsilon_H \otimes \epsilon_C$. Conditions (4) and (5) say that $\rho: H \otimes^s C \rightarrow C$ is a morphism of coalgebras.

Notations 1.11. Let $n, m \in \mathbb{N}$. Given a braided bialgebra H we define the maps:

- (1) $c_n^m: H^m \otimes H^n \rightarrow H^n \otimes H^m$, recursively by $c_1^1 := c$,

$$c_n^1 := (H \otimes c_{n-1}^1) \circ (c \otimes H^{n-1}),$$

$$c_m^n := (c_n^{m-1} \otimes H) \circ (H^{m-1} \otimes c_n^1).$$

- (2) $\text{sc}_n: H^{2n} \rightarrow H^{2n}$, recursively by $\text{sc}_1 := c$,

$$\text{sc}_n := (H \otimes \text{sc}_{n-1} \otimes H) \circ (c \otimes \cdots \otimes c).$$

Remark 1.12. the map c_n^m acts on each element $(h_1 \otimes \cdots \otimes h_m) \otimes (l_1 \otimes \cdots \otimes l_n)$ in $H^m \otimes H^n$ carrying the h_i 's to the right by means of reiterated applications of c and the map sc_n acts on each element $h_1 \otimes \cdots \otimes h_{2n}$ of H^{2n} carrying the h_i 's, with i odd, to the right by means of reiterated applications of c .

Example 1.13. Let H be a braided bialgebra and let $n \in \mathbb{N}$. Then H^n is a left H -braided module coalgebra, with

- comultiplication $\Delta_{H^n}: H^n \rightarrow H^n \otimes H^n$, defined by

$$\Delta_{H^n} := (H \otimes \text{sc}_{n-1} \otimes H) \circ (\Delta_H \otimes \cdots \otimes \Delta_H),$$

- counit $\epsilon \otimes \cdots \otimes \epsilon$ (n -times),
- transposition $c_n^1: H \otimes H^n \rightarrow H^n \otimes H$,
- action $\rho: H \otimes H^n \rightarrow H^n$ defined by

$$h \cdot (h_1 \otimes \cdots \otimes h_n) = (hh_1) \otimes h_2 \otimes \cdots \otimes h_n.$$

Note that $(c_n^n \otimes H) \circ (H^n \otimes c_n^1) \circ (c_n^1 \otimes H^n) = (H^n \otimes c_n^1) \circ (c_n^1 \otimes H^n) \circ (H \otimes c_n^n)$.

1.2. The commutative algebra of central maps.

Definition 1.14. A braided coalgebra (C, ς) is a coalgebra C endowed with a bijective map $\varsigma: C \otimes C \rightarrow C \otimes C$ that satisfies the braided equation and that is compatible with the coalgebra structure of C . We call ς the *braid* of C . Let (C, ς) be a braided coalgebra. We say that ς is *involutive* if $\varsigma^2 = \text{id}_C$. If also $\varsigma \circ \Delta_C = \Delta_C$, then (C, ς) is said to be *cocommutative*.

Recall from [B-M] that an entwining structure (C, A, ψ) consists of a coalgebra C , an algebra A and a bijective map $\psi: C \otimes A \rightarrow A \otimes C$, which is compatible with the coalgebra structure of C and the algebra structure of A . Assume that (C, ς) is a braided coalgebra. We say that the entwining structure (C, A, ψ) is *compatible with ς* or simply that (C, ς, A, ψ) is an *entwining structure* if

$$(A \otimes \varsigma) \circ (\psi \otimes C) \circ (C \otimes \psi) = (\psi \otimes C) \circ (C \otimes \psi) \circ (\varsigma \otimes A).$$

Example 1.15. Let H be a braided Hopf algebra, A an algebra and s a transposition of H on A . Then (H^n, c_n^n, A, s^n) , where $s^n: H^n \otimes A \rightarrow A \otimes H^n$ is recursively defined by $s^1 := s$ and $s^n := (s^{n-1} \otimes H) \circ (H^{n-1} \otimes s)$, is an entwining structure.

Definition 1.16. Let (C, ς, A, ψ) be an entwining structure. A map $f: C \rightarrow A$ is said to be *compatible with ψ* if $\psi \circ (C \otimes f) = (f \otimes C) \circ \varsigma$.

Remark 1.17. Let H be a braided bialgebra and let (H^n, c_n^n, A, s^n) be the entwining structure introduced in Example 1.15. We will say that a map $f: H^n \rightarrow A$ is *compatible with s* if $s \circ (H \otimes f) = (f \otimes H) \circ c_n^1$. It is easy to see that f is compatible with s if and only if f is compatible with s^n in the sense of Definition 1.16.

Definition 1.18. Let (C, ς, A, ψ) be an entwining structure. A map $f: C \rightarrow A$ is said to be *ψ -central* if $\mu_A \circ (A \otimes f) \circ \psi = \mu_A \circ (f \otimes A)$.

Let C be a coalgebra and A an algebra. Recall that $\text{Hom}_k(C, A)$ is an associative algebra with unit $\eta_A \circ \epsilon_C$ via the convolution product $f * g = \mu_A \circ (f \otimes g) \circ \Delta_C$.

Remark 1.19. Let (C, ς, A, ψ) be an entwining structure and let $f, g: C \rightarrow A$ be maps. If f is compatible with ψ and g is ψ -central, then $g * f = \mu_A \circ (f \otimes g) \circ \varsigma \circ \Delta_C$.

Proposition 1.20. Let (C, ς, A, ψ) be an entwining structure. The following assertions hold:

- (1) The set of all the maps from C to A which are compatible with ψ form a subalgebra of $\text{Hom}_k(C, A)$.

- (2) The set of all the maps from C to A which are compatible with ψ and ψ -central form a subalgebra of $\text{Hom}_k(C, A)$.

Proof. 1) This follows from the equalities

$$\begin{array}{c} \text{Diagram showing two equivalent ways to contract a vertical line with a } f \text{-bubble and a } g \text{-bubble.} \\ \text{Left: } \text{Diagram with } f \text{ and } g \text{ bubbles. Middle: } \text{Diagram with } f \text{ and } g \text{ bubbles. Right: } \text{Diagram with } (f \circ g) \text{ bubble.} \end{array} = = = \begin{array}{c} \text{Diagram with } (f \circ g) \text{ bubble.} \end{array}$$

2) By Remark 1.19 it suffices to note that

$$\begin{array}{c} \text{Diagram showing a sequence of six diagrams connected by equals signs.} \\ \text{Leftmost: } \text{Diagram with } f \text{ and } g \text{ bubbles. Middle: } \text{Diagram with } f \text{ and } g \text{ bubbles. Rightmost: } \text{Diagram with } (f \circ g) \text{ bubble.} \end{array} = = = = = = = \begin{array}{c} \text{Diagram with } (f \circ g) \text{ bubble.} \end{array}$$

□

Notation 1.21. We let $\text{Hom}_k^\psi(C, A)$ denote the subalgebra of $\text{Hom}_k(C, A)$ consisting of all the maps from C to A which are compatible with ψ and ψ -central. Note that if (C, ς) is cocommutative, then $\text{Hom}_k^\psi(C, A)$ is commutative.

Let (C, ς, A, ψ) be an entwining structure and let f be a convolution invertible element in $\text{Hom}_k(C, A)$. Assume that (C, ς) is cocommutative. Next we will prove that if f is compatible with ψ and ψ -central, then f^{-1} is so too. To carry out this task we will need the following result (see [Mo, Pag. 91]).

Lemma 1.22. Let A be an algebra and C a coalgebra. Let $\text{End}_A^C(C \otimes A)$ be the k -algebra of all right A -linear and left C -colinear endomorphisms of $C \otimes A$. The map $T_A^C: \text{Hom}_k(C, A) \rightarrow \text{End}_A^C(C \otimes A)$, given by $T_A^C(g)(c \otimes a) = c_{(1)} \otimes g(c_{(2)})a$, is an isomorphism of algebras (here $\text{Hom}_k(C, A)$ is considered as an algebra via the convolution product and $\text{End}_A^C(C \otimes A)$ is considered as an algebra via the composition of endomorphisms). The inverse map of T_A^C is given by $(T_A^C)^{-1}(g)(c) = (\epsilon \otimes A) \circ g(c \otimes 1)$.

Let $f \in \text{Hom}_k(C, A)$. It is easy to see that f is compatible with ψ if and only if

$$(C \otimes \psi) \circ (\varsigma \otimes A) \circ (C \otimes T_A^C(f)) = (T_A^C(f) \otimes C) \circ (C \otimes \psi) \circ (\varsigma \otimes A).$$

Theorem 1.23. Let $f \in \text{Hom}_k(C, A)$ be a convolution invertible element. If f is compatible with ψ and ψ -central, then f^{-1} is also.

Proof. Let g be the convolution inverse of f . The fact that g is compatible with ψ it follows immediately from the above comment. To see that it is ψ -central it is sufficient to check that

$$\begin{array}{c} \text{Diagram showing two equivalent ways to contract a vertical line with a } g \text{-bubble.} \\ \text{Left: } \text{Diagram with } g \text{ bubble. Middle: } \text{Diagram with } g \text{ bubble. Right: } \text{Diagram with } g \text{ bubble.} \end{array} = = = \begin{array}{c} \text{Diagram with } g \text{ bubble.} \end{array}$$

But this follows immediately from Lemma 1.22 and the fact that

The diagram consists of six boxes arranged horizontally, each containing a complex web of strands labeled C and A . The strands represent the components of the convolution operation. The first two boxes show the expression $(f * g) * f$, while the next four boxes show the expression $f * (g * f)$. The strands are colored blue and red, and circular nodes represent convolutional operations. The final box shows the result $\text{id}_{C \otimes A}$.

where the first equality follows from the fact that f is ψ -central, the second one from the compatibility of ψ with μ_A , the third one from the coassociativity of Δ_C , the associativity of μ_A and the fact that g is compatible with ψ , the fourth one from the compatibility of ψ with Δ_C , the fifth one from the cocommutativity of (C, ς) and the sixth one from the fact that g is the convolution inverse of f . \square

Let (C, ς, A, ψ) be an entwining structure. We let $\text{Reg}^\psi(C, A)$ denote the group of units of $\text{Hom}_k^\psi(C, A)$. Note that if (C, ς) is cocommutative, then, By remark in Notation 1.21 and Theorem 1.23, $\text{Reg}^\psi(C, A)$ is the abelian group made out of all the convolution invertible elements $f \in \text{Hom}_k^\psi(C, A)$.

Let H be a braided bialgebra and let (C, ς, A, ψ) be as above. Assume that we have a left H -braided module coalgebra structure (C, s_C) on C and a left H -braided module algebra structure (A, s_A) on A . Let $\text{Hom}_H^\psi((C, s_C), (A, s_A))$ be the set of all the elements $f \in \text{Hom}_k^\psi(C, A)$ that are H -linear maps and satisfy $s_A \circ (H \otimes f) = (f \otimes H) \circ s_C$. It is easy to see that $\text{Hom}_H^\psi((C, s_C), (A, s_A))$ is a subalgebra of $\text{Hom}_k^\psi(C, A)$. We define $\text{Reg}_H^\psi((C, s_C), (A, s_A))$ as the group of units of $\text{Hom}_H^\psi((C, s_C), (A, s_A))$.

It is immediate that $f \in \text{Hom}_H^\psi((C, s_C), (A, s_A))$ if and only if

$$s_{C \otimes A} \circ T_A^C(f) = T_A^C(f) \circ s_{C \otimes A}$$

and $T_A^C(f)$ is H -linear, where $C \otimes A$ is considered as a left H -module via the diagonal action. From this it follows that if $f \in \text{Hom}_H^\psi((C, s_C), (A, s_A))$ is convolution invertible, then $f^{-1} \in \text{Hom}_H^\psi((C, s_C), (A, s_A))$. So $\text{Reg}_H^\psi((C, s_C), (A, s_A))$ is the abelian group made out of all elements in $\text{Hom}_H^\psi((C, s_C), (A, s_A))$, which are the convolution invertible.

Next we consider the entwining structure (H^n, c_n^1, A, s^n) introduced in Examples 1.13 and 1.15.

Proposition 1.24. *Assume that H is a cocommutative braided bialgebra. Then $\text{Reg}_H^{s^n}((H^n, c_n^1), (A, s))$ is the commutative group of the convolution invertible H -linear maps $f: H^n \rightarrow A$ satisfying:*

- (1) $s \circ (H \otimes f) = (f \otimes H) \circ c_n^1$,
- (2) $\mu_A \circ (f \otimes A) = \mu_A \circ (A \otimes f) \circ s^n$.

Proof. It follows immediately from the above comments and Remark 1.17. \square

Since c_n^1 and s^n are constructed from the braid of H and s respectively, we will write $\text{Reg}_H^{s^n}(H^n, A)$ instead of $\text{Reg}_H^{s^n}((H^n, c_n^1), (A, s))$ and $\text{Hom}_H^{s^n}(H^n, A)$ instead of $\text{Hom}_H^{s^n}((H^n, c_n^1), (A, s))$. Moreover, we let $\text{Hom}_k^{s^n}(H^n, A)$ and $\text{Reg}^{s^n}(H^n, A)$ denote the algebra of k -linear maps from H^n to A satisfying conditions (1) and (2) of Proposition 1.24 and its group of units, respectively. It is easy to see that $\text{Hom}_k^{s^n}(H^n, A) = \text{Hom}_k^{s^n}(H^n, A)$ and of course $\text{Reg}^{s^n}(H^n, A) = \text{Reg}^{s^n}(H^n, A)$.

2. THE BRAIDED SWEEDLER COHOMOLOGY

In this section H will denote a cocommutative braided Hopf algebra. Let $\mathcal{B}(H)$ be the category whose objects are the left H -module coalgebras (H^n, c_n^1) with $n \geq 1$, and whose arrows are the maps of left H -module coalgebras

$$f: (H^n, c_n^1) \rightarrow (H^m, c_m^1)$$

such that $s^m \circ (f \otimes A) = (A \otimes f) \circ s^n$, for each left H -module algebra (A, s) . We have a simplicial complex in $\mathcal{B}(H)$ with objects $\{(H^{n+1}, c_{n+1}^1)\}_{n \geq 0}$ and face and degeneracy operators

$$\partial_i: (H^{n+1}, c_{n+1}^1) \rightarrow (H^n, c_n^1) \quad \text{and} \quad s_i: (H^{n+1}, c_{n+1}^1) \rightarrow (H^{n+2}, c_{n+2}^1)$$

given by

$$\begin{aligned} \partial_i(h_0 \otimes \cdots \otimes h_n) &= h_0 \otimes \cdots \otimes h_i h_{i+1} \otimes \cdots \otimes h_n \quad \text{for } i = 0, \dots, n-1, \\ \partial_n(h_0 \otimes \cdots \otimes h_n) &= h_0 \otimes \cdots \otimes h_{n-1} \epsilon(h_n) \end{aligned}$$

and

$$s_i(h_0 \otimes \cdots \otimes h_n) = h_0 \otimes \cdots \otimes h_i \otimes 1 \otimes h_{i+1} \otimes \cdots \otimes h_n \quad \text{for } i = 0, \dots, n.$$

Let (A, s) be a left H -module algebra. Let \mathcal{Ab} be the category of abelian groups. It is easy to see that $\text{Reg}_H^s(-, A): \mathcal{B}(H) \rightarrow \mathcal{Ab}$ is functorial. Applying this functor to the above simplicial complex, we obtain a cosimplicial complex. Following [Sw] we let ∂^i and s^i ($0 \leq i \leq n$) denote the coface operators

$$\text{Reg}_H^s(\partial_i, A): \text{Reg}_H^s(H^n, A) \rightarrow \text{Reg}_H^s(H^{n+1}, A)$$

and the codegenerations

$$\text{Reg}_H^s(s_i, A): \text{Reg}_H^s(H^{n+2}, A) \rightarrow \text{Reg}_H^s(H^{n+1}, A),$$

respectively. Let $d^{n-1}: \text{Reg}_H^s(H^n, A) \rightarrow \text{Reg}_H^s(H^{n+1}, A)$ be the map

$$d^{n-1} = \partial^0 * (\partial^1)^{-1} * \cdots * (\partial^n)^{\pm 1}.$$

The cochain complex

$$\text{Reg}_H^s(H, A) \xrightarrow{d^0} \text{Reg}_H^s(H^2, A) \xrightarrow{d^1} \text{Reg}_H^s(H^3, A) \xrightarrow{d^2} \cdots,$$

associated with the above cosimplicial complex, is called the *braided Sweedler cochain complex* of (A, s) . The *braided Sweedler cohomology* $\mathbf{H}^*(H, A, s)$, of H in (A, s) , is defined to be the cohomology of this complex. Let N_s^n be the subgroup of $\text{Reg}_H^s(H^{n+1}, A)$ defined by

$$N_s^n := \ker(s^0) \cap \cdots \cap \ker(s^{n-1}).$$

Note that $N_s^0 = \text{Reg}_H^s(H, A)$. By a well-known general result about cosimplicial complexes, $(N^*, d_{|N^*})$ is a subcomplex of $(\text{Reg}_H^s(H^{*+1}, A), d^*)$ (which we call the *braided Sweedler normalized cochain complex* of (A, s)) and the inclusion map, from $(N^*, d_{|N^*})$ to $(\text{Reg}_H^s(H^{*+1}, A), d^*)$, is a quasi-isomorphism.

Let $i_n: \text{Hom}_H(H^{n+1}, A) \rightarrow \text{Hom}_k(H^n, A)$ be the algebra isomorphism induced by the map $x \mapsto 1 \otimes x$ from H^n to H^{n+1} .

Lemma 2.1. *The map i_n induce an abelian group isomorphism*

$$\iota_n: \text{Reg}_H^s(H^{n+1}, A) \rightarrow \text{Reg}_H^s(H^n, A).$$

Proof. Let $f \in \text{Hom}_H(H^{n+1}, A)$. It suffices to show that

$$\begin{aligned} s \circ (H \otimes f) &= (f \otimes H) \circ c_{n+1}^1 \Leftrightarrow s \circ (H \otimes i(f)) = (i(f) \otimes H) \circ c_n^1, \\ \mu_A \circ (f \otimes A) &= \mu_A \circ (A \otimes f) \circ s^{n+1} \Leftrightarrow \mu_A \circ (i(f) \otimes A) = \mu_A \circ (A \otimes i(f)) \circ s^n. \end{aligned}$$

It is easy to check the first assertion and that

$$\mu_A \circ (f \otimes A) = \mu_A \circ (A \otimes f) \circ s^{n+1} \Rightarrow \mu_A \circ (i(f) \otimes A) = \mu_A \circ (A \otimes i(f)) \circ s^n.$$

Assume that $\mu_A \circ (i(f) \otimes A) = \mu_A \circ (A \otimes i(f)) \circ s^n$ and write $C = H^n$, $C' = H^{n+1}$ and $g = i(f)$. We have:

where the second equality follows from items (1) and (4) of Remark 1.8 and the facts that Δ is coassociative and compatible with c and g is compatible with s , the third one from the hypothesis, the fourth one from item (3) of Remark 1.8 and the facts that S is compatible with c and c is involutive, the fifth one follows from items (1) and (4) of Remark 1.8 and the facts that Δ is coassociative and compatible with c and the sixth one follows from the fact that S is compatible with s , that $(H \otimes S) \circ \Delta \circ S = (S \otimes H) \circ \Delta$ (since H is cocommutative), the coassociativity of Δ and item (1) of Remark 1.8. \square

The $\text{Reg}^s(H^n, A)$'s are the objects of a cosimplicial complex with coface operators

$$\delta^i: \text{Reg}^s(H^{n-1}, A) \rightarrow \text{Reg}^s(H^n, A) \quad i = 0, \dots, n$$

and codegenerations

$$\sigma^i: \text{Reg}^s(H^{n+1}, A) \rightarrow \text{Reg}^s(H^n, A) \quad i = 0, \dots, n,$$

defined by

$$\delta^i(f)(h_1 \otimes \dots \otimes h_n) = \begin{cases} h_1 \cdot f(h_2 \otimes \dots \otimes h_n) & \text{if } i = 0, \\ f(h_1 \otimes \dots \otimes h_i h_{i+1} \otimes \dots \otimes h_n) & \text{if } 0 < i < n, \\ f(h_1 \otimes \dots \otimes h_{n-1}) \epsilon(h_n) & \text{if } i = n, \end{cases}$$

and

$$\sigma^i(f)(h_1 \otimes \dots \otimes h_n) = f(h_1 \otimes \dots \otimes h_i \otimes 1 \otimes h_{i+1} \otimes \dots \otimes h_n),$$

respectively. Furthermore, the map $\iota_*: \text{Reg}_H^s(H^{*+1}, A) \rightarrow \text{Reg}^s(H^*, A)$ is an isomorphism of cosimplicial complexes. We let

$$\text{Reg}^s(k, A) \xrightarrow{D^0} \text{Reg}^s(H, A) \xrightarrow{D^1} \text{Reg}^s(H^2, A) \xrightarrow{D^2} \dots$$

denote the cochain complex associated with the cosimplicial complex $\text{Reg}^s(H^*, A)$. By definition $D^{n-1} = \delta^0 * (\delta^1)^{-1} * \dots * (\delta^n)^{\pm 1}$. So, $(\text{Reg}^s(H^*, A), D^*)$ gives the braided Sweedler cohomology of H in (A, s) . Of course this cohomology can be also compute by the normalized subcomplex $(\text{Reg}_+^s(H^*, A), D^*)$ of $(\text{Reg}^s(H^*, A), D^*)$.

Let ${}^sA := \{a \in A : s(h \otimes a) = a \otimes h \text{ for all } h \in H\}$. It is immediate that sA is a subalgebra of A . Notice that the map $f \mapsto f(1)$ is an isomorphism from $\text{Reg}^s(k, A)$ to $({}^sA \cap Z(A))^\times$. Let $a \in {}^sA \cap Z(A)$ be a regular element. By definition

$$D^0(a)(h) = (h_{(1)} \cdot a)\epsilon(h_{(2)})a^{-1} = (h \cdot a)a^{-1}.$$

Thus $a \in H^0(H, A, s)$ if and only if $h \cdot a = \epsilon(h)a$, and so

$$H^0(H, A, s) = ({}^sA \cap Z(A))^\times \cap {}^H A.$$

Next, we compute $H^1(H, A, s)$.

Definition 2.2. A map $f: H \rightarrow A$ is a *crossed homomorphism* if

$$f \circ \mu_H = \mu_A \circ (f \otimes \rho_A) \circ (\Delta \otimes f).$$

A crossed homomorphism f is called *inner* if there exists $a \in ({}^sA \cap Z(A))^\times$ so that $f = D^0(a)$.

Let $f \in \text{Reg}^s(H, A)$. It is easy to check that f is an 1-cocycle of the complex $(\text{Reg}^s(H^*, A), D^*)$ if and only if

$$f \circ \mu_H = \begin{array}{c} \text{Diagram showing two strands labeled } H \text{ crossing each other, with } f \text{ circles at the crossings.} \end{array} .$$

But, since H is cocommutative and f is s -central and compatible with s ,

$$\begin{array}{c} \text{Diagram showing two strands labeled } H \text{ crossing each other, with } f \text{ circles at the crossings.} \\ = \\ \text{Diagram showing two strands labeled } H \text{ crossing each other, with } f \text{ circles at the crossings.} \\ = \\ \text{Diagram showing two strands labeled } H \text{ crossing each other, with } f \text{ circles at the crossings.} \\ = \\ \text{Diagram showing two strands labeled } H \text{ crossing each other, with } f \text{ circles at the crossings.} \\ = \\ \text{Diagram showing two strands labeled } H \text{ crossing each other, with } f \text{ circles at the crossings.} \end{array} .$$

So, $H^1(H, A, s)$ is the group of the compatible with s and s -central regular crossed homomorphisms divided by the subgroup form by the inner crossed homomorphisms.

3. BRAIDED HOPF CROSSED PRODUCTS AND $H^2(H, A, s)$

Let H be a braided bialgebra and let (A, s) be a left H -module algebra. We let $\chi: H \otimes A \rightarrow A \otimes H$ denote the map defined by $\chi := (\rho \otimes H) \circ (H \otimes s) \circ (\Delta \otimes A)$, where $\rho: H \otimes A \rightarrow A$ is the action of H on A . Suppose given a map $f: H^2 \rightarrow A$. Let $\mathcal{F}_f: H^2 \rightarrow A \otimes H$ be the map defined by $\mathcal{F}_f := (f \otimes \mu) \circ \Delta_{H^2}$.

Definition 3.1 (G-G, Definition 9.2). We say that a map $f: H^2 \rightarrow A$ is *normal* if $f(1 \otimes x) = f(x \otimes 1) = \epsilon(x)$ for all $x \in H$, and that f is a *cocycle* that satisfies the *twisted module condition* if

$$\begin{array}{ccc} \text{Diagram showing two strands labeled } H \text{ crossing each other, with } f \text{ circles at the crossings.} & = & \text{Diagram showing two strands labeled } H \text{ crossing each other, with } f \text{ circles at the crossings.} \\ \text{and} & & \text{Diagram showing two strands labeled } H \text{ crossing each other, with } f \text{ circles at the crossings.} = \text{Diagram showing two strands labeled } H \text{ crossing each other, with } f \text{ circles at the crossings.} \end{array} , \quad \text{where } \Downarrow = f.$$

More precisely, the first equality is the cocycle condition and the second one is the twisted module condition.

From [G-G1, Section 9], we know that if $f: H^2 \rightarrow A$ is a normal cocycle compatible with s satisfying the twisted module condition, then $A \otimes H$ is an associative algebra with unit $1 \otimes 1$ via

$$\mu := (\mu_A \otimes H) \circ (\mu_A \otimes \mathcal{F}_f) \circ (A \otimes \chi \otimes H).$$

This algebra is called the *crossed product* of (A, s) with H associated with f , and denoted $A \#_f H$. The element $a \otimes h$ of $A \#_f H$ will be usually written $a \# h$. The cocycle f is said to be *invertible* if it invertible with respect to the convolution product in $\text{Hom}_k(H^2, A)$.

3.1. Equivalence of braided crossed products. In this Subsection we recall from [G-G1, Section 12] the notion of equivalence of crossed products. For this we need previously to recall the concept of right H -braided comodule algebra introduced in Section 5 of the same paper.

It is immediate that (H, c) is a coalgebra in \mathcal{LB}_H . Then we can considerer right (H, c) -comodules in \mathcal{LB}_H . We will refer to them as *right H -braided comodules* or simply as *right H -comodules*. For instance (k, τ) is a right H -comodule via the trivial coaction and the tensor product $(V, s_V) \otimes (U, s_U)$ of two right H -comodules is also via the codiagonal coaction. We let $(\mathcal{LB}_H)^H$ denote the category of right H -comodules. This is a monoidal category with the usual associativity and unit constraints. By definition a *right H -braided comodule algebra* (or simply a *right H -comodule algebra*) is an algebra in $(\mathcal{LB}_H)^H$. As above let (A, s) be a left H -module and let f be a normal cocycle compatible with s satisfying the condition of twisted module. The crossed $A \#_f H$ of (A, s) with H , associated with f , is a right H -comodule algebra when is endowed with the transposition $\hat{s} = s \otimes c$ and the coaction $A \otimes \Delta$.

Definition 3.2. We said that two crossed products $A \#_f H$ and $A \#_{f'} H$, of (A, s) with H , are equivalent if there is an isomorphism of H -comodule algebras

$$g: (A \#_f H, \hat{s}) \rightarrow (A \#_{f'} H, \hat{s}),$$

which is also an A -linear map.

Assume that H is a braided Hopf algebra. From [G-G1, Corollary 12.4] we know that $A \#_f H$ and $A \#_{f'} H$ are equivalent if and only if there exists a convolution invertible map $u: H \rightarrow A$ such that

- (1) $u(1) = 1$,
- (2) $(u \otimes H) \circ c = s \circ (H \otimes u)$,
- (3) $\rho = \mu_A \circ (\mu_A \otimes u) \circ (u^{-1} \otimes \chi) \circ (\Delta \otimes A)$,
- (4) $f' = \mu_A^2 \circ (A \otimes \rho \otimes \mu_A) \circ (u^{-1} \otimes H \otimes u^{-1} \otimes A \otimes u) \circ (\Delta \otimes H \otimes \mathcal{F}_f) \circ \Delta_{H^2}$.

Condition (1) is usually expressed saying that u is normal and condition (2) that u is compatible with s . Furthermore, since the right side of (4) is equal to

$$(u^{-1} \otimes \epsilon) * (\rho \circ (H \otimes u^{-1})) * f * (u \circ \mu_H),$$

this last condition is equivalent to

$$(4') [\rho \circ (H \otimes u)] * (u \otimes \epsilon) * f' = f * (u \circ \mu_H).$$

Let H be a cocommutative braided Hopf algebra and let (A, s) be a left H -module algebra. Our aim in this section is to show that $H^2(H, A, s)$ classifies the equivalence classes of crossed products $A \#_f H$, with f convolution invertible. The following results are well-known in the classical case (H a cocommutative Hopf algebra and s the flip). Their importance for our task is evident.

Proposition 3.3. *If H is a cocommutative braided Hopf algebra, then a map $u: H \rightarrow A$ satisfies condition (3) if and only if it is s -central.*

Proof. It is easy to see that ρ satisfies the equality in the statement if and only if

$$\text{Diagram showing two configurations of a loop with nodes } H \text{ and } A, \text{ connected by a vertical line with a loop. The left configuration has node } u \text{ at the bottom-left. The right configuration shows the loop moved to the top-left, and node } u \text{ is now at the bottom-right. An equals sign separates the two configurations.}$$

But since H is cocommutative and (A, s) is a left H -module algebra,

$$\begin{array}{c} H \\ \diagdown \\ \text{---} \\ \diagup \\ A \end{array} = \begin{array}{c} H \\ \diagup \\ \text{---} \\ \diagdown \\ A \end{array} = \begin{array}{c} H \\ \diagup \\ \text{---} \\ \diagup \\ A \end{array} .$$

The result follows now immediately using that $(H \otimes \rho) \circ (\Delta \otimes A)$ is bijective with inverse $(H \otimes \rho \circ S) \circ (\Delta \otimes A)$. \square

Proposition 3.4. *If H is a cocommutative braided Hopf algebra, then a map*

$$f: H^2 \rightarrow A$$

satisfies the twisted module condition if and only if it is s -central.

Proof. Since (A, s) is an H -module algebra and H is cocommutative, we have

$$\text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} = \text{Diagram 4} = \text{Diagram 5} = \text{Diagram 6}.$$

So, f satisfies the twisted module condition if and only if

$$\text{Diagram showing a relation between two configurations of strands and bubbles. The left side shows a crossing of strands with bubbles attached, followed by an equals sign. The right side shows a simplified configuration where the strands and bubbles are rearranged.}$$

But this happens if and only if f is s -central, since $(H^2 \otimes \rho \circ \mu_H) \circ (\Delta_{H^2} \otimes A)$ is bijective with inverse $(H^2 \otimes \rho \circ S \circ \mu_H) \circ (\Delta_{H^2} \otimes A)$. \square

Theorem 3.5. Assume that H is a cocommutative braided Hopf algebra. Then there is a bijective correspondence between $\mathsf{H}^2(H, A, s)$ and the equivalence classes of crossed products of (A, s) with H , whose cocycle is convolution invertible.

Proof. By Proposition 3.4, the elements of $\text{Reg}_+^s(H^2, A)$ are the convolution invertible normal maps, compatible with s , satisfying the twisted module condition. It is easy to see that an element $f \in \text{Reg}_+^s(H^2, A)$ is a cocycle in the sense of Definition 3.1 if and only if $(\delta^0 * \delta^2)(f) = (\delta^3 * \delta^1)(f)$. That is, if and only if f is a 2-cocycle of the complex $(\text{Reg}_+^s(H^*, A), D^*)$. It remains to check that two crossed products $A \#_f H$ and $A \#_{f'} H$ are equivalent if and only if $f * f'^{-1}$ is a coboundary in the complex $(\text{Reg}_+^s(H^*, A), D^*)$. By Proposition 3.3, we know that the elements of $\text{Reg}_+^s(H, A)$ are the convolution invertible normal maps, compatible with s , that

satisfy condition (3) in the discussion preceding Proposition 3.3. It is easy to see that there exists $u \in \text{Reg}_+^s(H, A)$ such that $(\delta^0 * \delta^2)(u) * f' = f * \delta^1(u)$ if and only if condition (4') is also satisfied. That is, if and only if $f * f'^{-1}$ is the coboundary of u in the complex $(\text{Reg}_+^s(H^*, A), D^*)$. \square

4. COMPARISON WITH A VARIANT OF GROUP COHOMOLOGY

Let G be a group. We will say that a transposition $s: k[G] \otimes A \rightarrow A \otimes k[G]$ is induced by an $\text{Aut}(G)^{\text{op}}$ -graduation $A = \bigoplus_{\zeta \in \text{Aut}(G)} A_\zeta$ of A if $s(g \otimes a) = a \otimes \zeta(g)$ for all $g \in G$ and $a \in A_\zeta$. For instance, by [G-G1, Theorem 4.14], if G is finitely generated, then each transposition of $k[G]$ on A is induced by an $\text{Aut}(G)^{\text{op}}$ -graduation on A , and this gradation is unique. Let sA be as in Section 2. It is easy to check that ${}^sA = A_{\text{id}}$. In this section, we show that if G is a group and (A, s) is a left $k[G]$ -module algebra, whose transposition s is induced by an $\text{Aut}(G)^{\text{op}}$ -graduation of A , then, the braided Sweedler cohomology of $k[G]$ in (A, s) coincide with a variation of the group cohomology of G with coefficients in the abelian group $Z(A_{\text{id}})^\times$ of units of the center of A_{id} . In order to make out this we first recall some well-known concepts and notations and we introduce other ones.

a: From [G-G1, Example 9.8], we know that the action of $k[G]$ on (A, s) satisfies

- (1) $g \cdot (ab) = (g \cdot a)(\zeta(g) \cdot b)$ if $a \in A_\zeta$ and $g \in G$,
- (2) $g \cdot 1 = 1$, for all $g \in G$,
- (3) $1 \cdot a = a$, for all $a \in A$,
- (4) $g \cdot a \in A_\zeta$, for all $g \in G$, $a \in A_\zeta$,

In particular, $k[G]$ acts on A_{id} in the classical sense. From this it follows easily that the action of $k[G]$ carry $Z(A_{\text{id}})^\times$ into itself.

b: The automorphism group $\text{Aut}(G)$ acts on G via $\zeta \otimes a \mapsto \zeta(g)$. Let

$$G \rtimes \text{Aut}(G)$$

be the associated semidirect product and let ${}_{G \rtimes \text{Aut}(G)} \text{Mod}$ be the category of left $k[G \rtimes \text{Aut}(G)]$ -modules. Let $F: {}_{G \rtimes \text{Aut}(G)} \text{Mod} \rightarrow \text{Ab}$ be the contravariant additive functor defined on objects and arrows, by

- $F(M)$ is the space of $k[G]$ -linear maps $\varphi: M \rightarrow Z(A_{\text{id}})^\times$, such that $\varphi(m)a = a\varphi(\zeta \cdot m)$ for all $m \in M$, $\zeta \in \text{Aut}(G)$ and $a \in A_\zeta$,
- $F(\alpha)(\varphi) := \varphi \circ \alpha$,

respectively.

Note that k is a left $k[G \rtimes \text{Aut}(G)]$ -module via the trivial action and that $k[G]^{n+1}$ is a left $k[G \rtimes \text{Aut}(G)]$ -module via

$$(g, \zeta) \cdot (g_0 \otimes \cdots \otimes g_n) = g\zeta(g_0) \otimes \zeta(g_1) \otimes \cdots \otimes \zeta(g_n),$$

for all $n \geq 0$. Moreover, each $k[G]^{n+1}$ is projective relative to the algebra extension $k[\text{Aut}(G)] \hookrightarrow k[G \rtimes \text{Aut}(G)]$.

Theorem 4.1. *Let $R^n F$ be the n -th right derived functor of F relative to the algebra extension $k[\text{Aut}(G)] \hookrightarrow k[G \rtimes \text{Aut}(G)]$. The n -th braided Sweedler cohomology group $H^n(k[G], A, s)$ is canonically isomorphic to $R^n F(k)$, for all $n \geq 0$.*

Proof. It is immediate that a map $\varphi: k \rightarrow A$ is s -central, compatible with s and convolution invertible if and only if $\text{Im}(\varphi) \subseteq (A_{\text{id}} \cap Z(A))^\times$. Assume that $n > 0$ and let $\varphi: k[G]^n \rightarrow A$. It is easy to check that:

- (1) φ is compatible with s if and only if $\text{Im}(\varphi) \subseteq A_{\text{id}}$.

- (2) φ is s -central if and only if $\varphi(g_1 \otimes \cdots \otimes g_n)a = a\varphi(\zeta(g_1) \otimes \cdots \otimes \zeta(g_n))$ for all $g_1, \dots, g_n \in G$, $\zeta \in \text{Aut}(G)$ and $a \in A_\zeta$.
- (3) φ is convolution invertible if and only if $\varphi(g_1 \otimes \cdots \otimes g_n) \in A^\times$ for all $g_1, \dots, g_n \in G$.

In particular this implies that $\text{Im}(\varphi) \subseteq Z(A_{\text{id}})^\times$. From these facts it follows easily that the map

$$\mathbb{J}: \text{Reg}^s(k[G]^n, A) \rightarrow F(k[G]^{n+1}),$$

defined by

$$\mathbb{J}(\varphi)(g_0 \otimes \cdots \otimes g_n) = g_0 \cdot \varphi(g_1 \otimes \cdots \otimes g_n) \text{ for all } g_0, \dots, g_n \in G,$$

is an isomorphism. So, by transporting of structure, we obtain a cochain complex isomorphic to $(\text{Reg}^s(k[G]^*, A), D^*)$, whose n -th cochain group is $F(k[G]^{n+1})$. Consider now the non normalized Barr resolution $B_*^u(G)$ of k as a left $k[G]$ -module. It is immediate that the canonical contraction homotopy of $B_*^u(G) \rightarrow k$ is $k[\text{Aut}(G)]$ -linear. Since each $k[G]^{n+1}$ is projective relative to the algebra extension $k[\text{Aut}(G)] \hookrightarrow k[G \rtimes \text{Aut}(G)]$, to finish the proof it suffices to notice that applying F to $B_*^u(G)$ one obtain the same complex as before. \square

Example 4.2. If s is the flip, or (which is equivalent) $A_\zeta = 0$ for all $\zeta \neq \text{id}$, then $F(B_*^u(G))$ is the canonical non normalized complex computing the group cohomology of k with coefficients in $Z(A)^\times$. So, Theorem 4.1 generalizes [Sw, Theorem 3.1]

Remark 4.3. Assume that $Z(A_{\text{id}}) \subseteq Z(A)$ and that A is strongly $\text{Aut}(G)^{\text{op}}$ -graded. So, for each $\zeta \in \text{Aut}(G)$, there exist $a_1, \dots, a_l \in A_\zeta$ and $b_1, \dots, b_l \in A_{\zeta^{-1}}$ such that $\sum a_i b_i = 1$. If $\varphi: k[G]^{n+1} \rightarrow Z(A)_{\text{id}}$ is s -central, then, for each $g_0, \dots, g_n \in G$ and $\zeta \in \text{Aut}(G)$, we have

$$\begin{aligned} \varphi(g_0 \otimes \cdots \otimes g_n) &= \sum_{i=1}^l \varphi(g_0 \otimes \cdots \otimes g_n) a_i b_i \\ &= \sum_{i=1}^l a_i \varphi(\zeta(g_0) \otimes \cdots \otimes \zeta(g_n)) b_i \\ &= \varphi(\zeta(g_0) \otimes \cdots \otimes \zeta(g_n)), \end{aligned}$$

where the second equality follows from the fact that φ is s -central and the third one from the fact that $\varphi(\zeta(g_0) \otimes \cdots \otimes \zeta(g_n)) \in Z(A)$. Conversely, if φ satisfies the above equality, then φ is s -central.

5. COMPARISON WITH A VARIANT OF LIE COHOMOLOGY

In this section k is a characteristic zero field, H is the enveloping algebra of a lie algebra L and (A, s) is a left H -module algebra. Using that the braid of H is the flip and H is cocommutative, it is easy to check that ${}^s A$ and its center $Z({}^s A)$ are left H -module algebras and that $f: H^n \rightarrow A$ is compatible with s if and only if $\text{Im}(f) \subseteq {}^s A$. Assume that f is also s -central. Then, $\text{Im}(f)$ is included in $Z({}^s A)$.

For each $n \geq 0$, let

$$C_s^n := \{f \in \text{Hom}_k^s(H^n, A) : f(x_1 \otimes \cdots \otimes x_n) = 0 \text{ if some } x_i \in k\}.$$

Let $\delta^{n+1}: C_s^n \rightarrow C_s^{n+1}$ be the map defined by

$$\begin{aligned} \delta^{n+1}(f)(x_1 \otimes \cdots \otimes x_{n+1}) &= x_1 \cdot f(x_2 \otimes \cdots \otimes x_{n+1}) \\ &\quad + \sum_{i=1}^n (-1)^i f(x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1}) \\ &\quad + (-1)^{n+1} f(x_1 \otimes \cdots \otimes x_n) \epsilon(x_{n+1}). \end{aligned}$$

It is easy to check that (C_s^*, δ^*) is a cochain complex. Indeed, it is immediate that (C_s^*, δ^*) is a subcomplex of the normalized Hochschild cochain complex $(C^*(H, Z(sA)^+), \delta^*)$ of H with coefficients in $Z(sA)^+$, where $Z(sA)^+$ is $Z(sA)$, endowed with the H -bimodule structure defined by $h \cdot a \cdot l = h\epsilon(l) \cdot a$. The n -th cohomology group of (C_s^*, δ^*) will be denoted $H_+^n(H, A, s)$.

Let $\tau: H \otimes Z(sA) \rightarrow Z(sA) \otimes H$ be the flip. In the proof of [Sw, Theorem 4.1], Sweedler shows that, for each $n \geq 1$, the map

$$\exp: C^n(H, Z(sA)^+) \rightarrow \text{Reg}_+^\tau(H^n, Z(sA)),$$

defined by $\exp(f) = \sum_{i=0}^{\infty} \frac{1}{i!} f^i$, where f^i denotes the i -th convolution power of f , is bijective and commutes with the coboundary maps. The inverse is the map $\log(g) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{1}{i} (g - e)^i$, where e is the unit of $\text{Reg}_+^\tau(H^n, Z(sA))$. Using that $\text{Hom}_k^s(H^n, A)$ is a subalgebra of $\text{Hom}_k(H^n, A)$, it is easy to see that \exp induce an isomorphism from C_s^n to $\text{Reg}_+^s(H^n, A, s)$. So we have the following result:

Theorem 5.1. $H^n(H, A, s) = H_+^n(H, A, s)$ for each $n \geq 2$.

As usual, we let $_H\mathcal{M}_H$ denote the category of all the H -bimodules and we let $\mathcal{V}\text{ect}$ denote the category of all the k -vector spaces.

Definition 5.2. Let (A, s) be a left H -module algebra and let M be a k -vector space. A map $\phi: M \otimes A \rightarrow A \otimes M$ is a *transposition* of M on (A, s) if it satisfies the following conditions:

(1) ϕ is compatible with the algebra structure of A . That is,

$$\begin{array}{ccc} \text{Diagram: } & & \text{Diagram: } \\ \text{Left: } & = & \text{Right: } \\ \text{Diagram showing compatibility with } A \text{ structure} & & \text{Diagram showing compatibility with } A \text{ structure} \end{array} \quad \text{and} \quad \begin{array}{c} \text{Diagram: } \\ \text{Left: } \\ \text{Diagram showing compatibility with } M \text{ structure} \\ \text{Right: } \\ \text{Diagram showing compatibility with } M \text{ structure} \end{array}.$$

(2) ϕ is compatible with the left action of H on A . That is,

$$\begin{array}{ccc} \text{Diagram: } & & \text{Diagram: } \\ \text{Left: } & = & \text{Right: } \\ \text{Diagram showing compatibility with } H \text{ action on } A \\ \text{where } \times \text{ is the flip.} & & \text{Diagram showing compatibility with } H \text{ action on } A \\ \text{where } \times \text{ is the flip.} & & \end{array}$$

(3) The following equalities hold:

$$\begin{array}{ccc} \text{Diagram: } & & \text{Diagram: } \\ \text{Left: } & = & \text{Right: } \\ \text{Diagram showing equality of two crossed terms} & & \text{Diagram showing equality of two crossed terms} \end{array}.$$

When $M \in {}_H\mathcal{M}_H$ is also required that

(4) ϕ is compatible with the right action of H on M . That is,

$$\begin{array}{ccc} \text{Diagram: } & & \text{Diagram: } \\ \text{Left: } & = & \text{Right: } \\ \text{Diagram showing compatibility with } H \text{ action on } M \\ \text{where } \times \text{ is the flip.} & & \text{Diagram showing compatibility with } H \text{ action on } M \\ \text{where } \times \text{ is the flip.} & & \end{array}$$

(5) ϕ be compatible with the left action of H on M . That is,

$$\begin{array}{ccc} \text{Diagram: } & & \text{Diagram: } \\ \text{Left: } & = & \text{Right: } \\ \text{Diagram showing compatibility with } H \text{ action on } M \\ \text{where } \times \text{ is the flip.} & & \text{Diagram showing compatibility with } H \text{ action on } M \\ \text{where } \times \text{ is the flip.} & & \end{array}$$

The pairs (M, ϕ) , consisting of an H -bimodule M and a transposition ϕ of M on (A, s) are the objects of a category $\mathfrak{T}_A^s(H\mathcal{M}_H)$, called the category of *transpositions* of H -bimodules on (A, s) . A *morphism* of transpositions $f: (M, \phi) \rightarrow (N, \varphi)$ is a left and right H -linear map $f: M \rightarrow N$, such that $\varphi \circ (f \otimes A) = (A \otimes f) \circ \phi$. In a similar way we define the category $\mathfrak{T}_A^s(\mathcal{V}ect)$ of *transpositions* of k -vector spaces on (A, s) . It is easy to check that both categories are abelian. Moreover, both are tensor categories:

- The unit of $\mathfrak{T}_A^s(\mathcal{V}ect)$ and $\mathfrak{T}_A^s(H\mathcal{M}_H)$ is (k, τ) , where $\tau: k \otimes A \rightarrow A \otimes k$ is the flip and k is endowed with the trivial module structure.
- Given (M, ϕ) and (N, φ) in $\mathfrak{T}_A^s(\mathcal{V}ect)$, the tensor product $(M, \phi) \otimes (N, \varphi)$ is the pair $(M \otimes N, \phi \otimes \varphi)$, where $\phi \otimes \varphi := (\phi \otimes N) \circ (M \otimes \varphi)$. If (M, ϕ) and (N, φ) belongs to $\mathfrak{T}_A^s(H\mathcal{M}_H)$, then $M \otimes N$ is also endowed with the left and right actions $h \cdot (m \otimes n) := h \cdot m \otimes n$ and $(m \otimes n) \cdot h := m \otimes n \cdot h$.

Let (M, ϕ) be an object in $\mathfrak{T}_A^s(H\mathcal{M}_H)$. An H bimodule map $f: M \rightarrow Z(^sA)^+$ is said to be ϕ -central if $\mu_A \circ (f \otimes A) = \mu_A \circ (A \otimes f) \circ \phi$. Let $\Xi: \mathfrak{T}_A^s(H\mathcal{M}_H) \rightarrow \mathcal{V}ect$ be the contravariant additive functor defined on objects and arrows by

- $\Xi(M, \phi)$ is the k -vector space, consisting of the H -bimodule maps f from M to $Z(^sA)^+$ which are ϕ -central,
- $\Xi(\alpha)(f) := f \circ \alpha$,

respectively.

Theorem 5.3. *Let $R^n\Xi$ be the n -th right derived functor of Ξ , relative to the class of the epimorphisms in $\mathfrak{T}_A^s(H\mathcal{M}_H)$ that split in $\mathfrak{T}_A^s(\mathcal{V}ect)$. The n -th cohomology group $H_+^n(H, A, s)$ of (C_s^*, δ^*) is canonically isomorphic to $R^n\Xi(H, s)$, for all $n \geq 0$.*

Proof. For each $n \geq 0$, let $\bar{s}_n: H \otimes \overline{H}^{\otimes n} \otimes H \otimes A \rightarrow A \otimes H \otimes \overline{H}^{\otimes n} \otimes H$ be the transposition of $H \otimes \overline{H}^{\otimes n} \otimes H$ on (A, s) , induced by s^{n+2} . Let $(H \otimes \overline{H}^{\otimes *}, b')$ be the canonical normalized resolution of H as an H -bimodule. It is easy to check that $((H \otimes \overline{H}^{\otimes *}, b'), \bar{s}_*)$ is a complex in $\mathfrak{T}_A^s(H\mathcal{M}_H)$ and that

$$H \xleftarrow{\mu} H \otimes H \xleftarrow{b'} H \otimes \overline{H} \otimes H \xleftarrow{b'} H \otimes \overline{H}^{\otimes 2} \otimes H \xleftarrow{b'} \dots,$$

where μ is defined by $\mu(h \otimes l) = hl$, is contractible as a complex in $\mathfrak{T}_A^s(\mathcal{V}ect)$. Moreover, it is immediate that $(H \otimes \overline{H}^{\otimes n}, \bar{s}_n)$ is relative projective for all n . To finish the proof it suffices to note that applying the functor Ξ to the resolution $((H \otimes \overline{H}^{\otimes *}, b'), \bar{s}_*)$ one obtain the cochain complex (C_s^*, δ^*) . \square

Corollary 5.4. $H^n(H, A, s)$ is isomorphic to $R^n\Xi(H, s)$, for all $n \geq 2$.

The following results will be useful to perform explicit computations.

Proposition 5.5. *Let (M, ϕ) be an object in $\mathfrak{T}_A^s(H\mathcal{M}_H)$ and let $f: M \rightarrow Z(^sA)^+$ an H -bimodule map. Assume that M is generated as an H -bimodule by $(m_i)_{i \in I}$. If $f(m_i)a = \mu_A \circ (A \otimes f) \circ \phi(m_i \otimes a)$ for all $i \in I$ and $a \in A$, then f is ϕ -central.*

Proof. It suffices to check that if $f(x)a = \mu_A \circ (A \otimes f) \circ \phi(x \otimes a)$, then

$$f(x \cdot h)a = \mu_A \circ (A \otimes f) \circ \phi(x \cdot h \otimes a) \quad \text{and} \quad f(h \cdot x)a = \mu_A \circ (A \otimes f) \circ \phi(h \cdot x \otimes a),$$

for all $h \in H$. The first equation is easy, since the right action of H on $\mathbb{Z}(^sA)^+$ is the trivial one. Let us prove the second one. We have:

where the first equality follows from the fact that f is left H -linear, the second one from the fact that (A, s) is an H -module algebra, the third one from the fact that $\text{Im}(f) \subseteq {}^sA$, the fourth one from the hypothesis, the fifth one from item (2) of Definition 5.2, the sixth one from the fact H is cocommutative and (A, s) is a left H -module algebra and the seventh one from the fact that f is left H -linear and from item (5) of Definition 5.2. \square

Proposition 5.6. *Let (M, ϕ) be an object in $\mathfrak{T}_A^s(H\mathcal{M}_H)$, let $f: M \rightarrow \mathbb{Z}(^sA)^+$ be an H -bimodule map and let V be a vector subspace of M such that $\phi(V \otimes A) \subseteq A \otimes V$. The set $A_{f,V}$, of all $a \in A$ satisfying $f(v)a = \mu_A \circ (A \otimes f) \circ \phi(v \otimes a)$ for all $v \in V$, is a subalgebra of A .*

Proof. It is immediate that $1 \in A_{f,V}$ and that $A_{f,V}$ is closed under addition. Let us check it is also closed under multiplication. Let $a, b \in A_{f,V}$. We have:

as we want. \square

6. THE CHEVALLEY-EILENBERG RESOLUTION

As in Section 5 let k be a characteristic zero field, H the enveloping algebra of a Lie algebra L and (A, s) a left H -module algebra. Our purpose is to show that in order to compute $H^*(H, A, s)$ it is possible to use a Chevalley-Eilenberg resolution type of H as an H -bimodule. For this we are going to show that this resolution is a complex in $\mathfrak{T}_A^s(H\mathcal{M}_H)$, which is contractible as a complex in $\mathfrak{T}_A^s(\mathcal{V}\text{ect})$, in a natural way.

6.1. A simple resolution. Consider three copies Y_L , Z_L and E_L of L . We will let Y_x , Z_x and e_x ($x \in L$) denote the elements of Y_L , Z_L and E_L respectively. So, the maps $x \mapsto Y_x$, $x \mapsto Z_x$ and $x \mapsto E_x$ will be isomorphisms of vector spaces. We assign degree 0 to Y_x and Z_x and degree 1 to E_x . Let (D_*, d_*) be the differential graded algebra generated by $Y_L \oplus Z_L \oplus E_L$ and the relations

- (1) $Y_{x'}Y_x = Y_xY_{x'} + \frac{1}{2}Y_{[x',x]_L}$, for $x', x \in L$,
- (2) $T_{x'}Y_x = Y_xT_{x'} + \frac{1}{2}T_{[x',x]_L}$, for $x', x \in L$,
- (3) $T_{x'}T_x = T_xT_{x'}$, for $x', x \in L$,
- (4) $e_xY_x = Y_xe_x + \frac{1}{2}e_{[x',x]_L}$, for $x', x \in L$,
- (5) $e_{x'}T_x = T_xe_{x'}$, for $x', x \in L$,
- (6) $e_x^2 = 0$, for $x', x \in L$,

where $T_x = Y_x - Z_x$; with differential defined by $d_1(e_x) = T_x$ for $x \in L$. Note that (D_*, d_*) is a complex of H -bimodules if we define $x \cdot W = Y_x W$ and $W \cdot x = W Z_x$ for $x \in L$. Note that $D_* = T_k(Y_L \oplus Z_L \oplus E_L)/R$, where R is the two sided ideal generated by the relations (1) – (6). Also note that from these relations it follows that

$$\begin{aligned} d_n(e_{x_1} \cdots e_{x_n}) &= \sum_{i=1}^n (-1)^{i-1} Y_{x_i} e_{x_1} \cdots \widehat{e_{x_i}} \cdots e_{x_n} \\ &\quad - \sum_{i=1}^n (-1)^{i-1} e_{x_1} \cdots \widehat{e_{x_i}} \cdots e_{x_n} Z_{x_i} \\ &\quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} e_{[x_i, x_j]} e_{x_1} \cdots \widehat{e_{x_i}} \cdots \widehat{e_{x_j}} \cdots e_{x_n}, \end{aligned}$$

where, as usual, the symbol $\widehat{e_{x_i}}$ means that the factor e_{x_i} is omitted.

We now introduce a new gradation p on D_* by defining

$$p(Y_i) = 0 \quad \text{and} \quad p(T_i) = p(e_i) = 1.$$

Let $\gamma_{*+1}: D_* \rightarrow D_{*+1}$ be the degree one derivation defined by

$$\gamma(Y_x) = 0, \quad \gamma(T_x) = e_x \quad \text{and} \quad \gamma(e_x) = 0.$$

It is easy to see that

$$(1) \quad (\gamma \circ d + d \circ \gamma)(P) = p(P)P \quad \text{for } P \text{ a } p\text{-homogeneous element.}$$

Let $\sigma_0: H \rightarrow D_0$ be the algebra map defined by $\sigma_0(x) = Y_x$ and, for $n \geq 0$, let $\sigma_{n+1}: D_n \rightarrow D_{n+1}$ be the map defined by

$$\sigma_{n+1}(P) = \begin{cases} \frac{1}{p(P)} \gamma(P) & \text{if } p(P) > 0, \\ 0 & \text{if } p(P) = 0. \end{cases}$$

From (1) it follows easily that σ is a left H -linear contracting homotopy of

$$(2) \quad H \xleftarrow{\mu} D_0 \xleftarrow{d_1} D_1 \xleftarrow{d_2} D_2 \xleftarrow{d_3} D_3 \xleftarrow{d_4} \cdots,$$

where μ is the H -bimodule map defined by $\mu(1) = 1$.

We are going to show that there are transpositions $s_{D_n}: D_n \otimes A \rightarrow A \otimes D_n$ such that (2), endowed with them, is a complex in $\mathfrak{T}_A^s(H\mathcal{M}_H)$ which is contractible as a complex in $\mathfrak{T}_A^s(\mathcal{V}ect)$, and such that each (D_n, s_{D_n}) is projective relative to the family of all epimorphism in $\mathfrak{T}_A^s(H\mathcal{M}_H)$ which split in $\mathfrak{T}_A^s(\mathcal{V}ect)$. In order to carry out our task we need first to describe the transposition $s: H \otimes A \rightarrow A \otimes H$ in terms of a basis $(x_i)_{i \in I}$ of L . For the sake of simplicity we assume that L is finite dimensional and $I = \{1, \dots, r\}$ (however it is possible to work without this restriction).

Let $\alpha_i^j: A \rightarrow A$ be the maps defined by $s(x_i \otimes a) = \sum_{j=1}^r \alpha_i^j(a) \otimes x_j$ and let $\bar{\alpha}$ be the matrix.

$$\bar{\alpha} = \begin{pmatrix} \alpha_1^1 & \dots & \alpha_1^r \\ \vdots & \ddots & \vdots \\ \alpha_r^1 & \dots & \alpha_r^r \end{pmatrix}$$

From [G-G3, Example 2.1.9], we know that the maps α_i^j satisfy

- (a) $\bar{\alpha}(1) = \text{id}$,
- (b) $\bar{\alpha}(ab) = \bar{\alpha}(a)\bar{\alpha}(b)$,
- (c) $\alpha_i^l \circ \alpha_j^m = \alpha_j^m \circ \alpha_i^l$, for all i, j, l, m ,
- (d) $s([x_i, x_j]_L \otimes a) \sum_{l < m} \alpha_i^l \circ \alpha_j^m(a) - \alpha_i^m \circ \alpha_j^l(a) \otimes [x_l, x_m]_L$.

Furthermore, using that s is bijective, it is easy to check that $\overline{\alpha} \in \text{GL}_r(\text{End}_k(A))$.

Since the maps α_i^j satisfy conditions (a), (b) and (c), it follows from [G-G3, Example 2.1.8] that there exists a unique transposition

$$s_T: T_k(Y_L \oplus Z_L \oplus E_L) \otimes A \rightarrow A \otimes T_k(Y_L \oplus Z_L \oplus E_L)$$

such that

$$\begin{aligned} s_T(Y_i \otimes A) &= \sum_{j=1}^r \alpha_i^j(a) \otimes Y_j, \\ s_T(Z_i \otimes A) &= \sum_{j=1}^r \alpha_i^j(a) \otimes Z_j, \\ s_T(e_i \otimes A) &= \sum_{j=1}^r \alpha_i^j(a) \otimes e_j, \end{aligned}$$

where $Y_i = Y_{x_i}$, $Z_i = Z_{x_i}$ and $e_i = e_{x_i}$, for all $i \in I$. Since conditions (c) and (d) imply that $s_T(R) = R$, the map s_T induces a transposition

$$s_D: D \otimes A \rightarrow A \otimes D.$$

Clearly $S_D(D_n \otimes A) = A \otimes D_n$ for each $n \geq 0$. Let $s_{D_n}: D_n \otimes A \rightarrow A \otimes D_n$ be the map induced by s_D . It is easy to check that $(D_n, s_{D_n}) \in \mathfrak{T}_A^s(H\mathcal{M}_H)$, d_n is an arrow in $\mathfrak{T}_A^s(H\mathcal{M}_H)$, σ_n is an arrow in $\mathfrak{T}_A^s(\text{Vect})$ and (D_n, s_{D_n}) is projective relative to the family of all epimorphism in $\mathfrak{T}_A^s(H\mathcal{M}_H)$ which split in $\mathfrak{T}_A^s(\text{Vect})$.

Theorem 6.1. *For $n \geq 2$ the Sweedler cohomology $H^n(H, A, s)$ is the cohomology of the cochain complex obtained applying the functor Ξ , introduced above Theorem 5.3, to the Chevalley-Eilenberg resolution $((D_*, s_{D_*}), d_*)$.*

Proof. It follows from the above discussion and Theorems 5.1 and 5.3. \square

6.2. Comparison maps. The H -bimodule morphisms $\varphi_n: D_n \rightarrow H \otimes \overline{H}^n \otimes H$ and $\phi_n: H \otimes \overline{H}^n \otimes H \rightarrow D_n$, recursively defined by

$$\begin{aligned} \varphi_0(1) &= 1 \otimes 1, \\ \varphi_n(e_{i_1} \cdots e_{i_n}) &= 1 \otimes \varphi_{n-1} \circ d_n(e_{i_1} \cdots e_{i_n}), \\ \phi_0(1 \otimes 1) &= 1, \\ \phi_n(1 \otimes P_1 \otimes \cdots \otimes P_n \otimes 1) &= \sigma_n \circ \phi_{n-1} \circ b'_n(1 \otimes P_1 \otimes \cdots \otimes P_n \otimes 1), \end{aligned}$$

are chain complexes morphisms from $((D_*, s_{D_*}), d_*)$ to $((H \otimes \overline{H}^{\otimes *}, b'))$ and from $((H \otimes \overline{H}^{\otimes *}, b'))$ to $((D_*, s_{D_*}), d_*)$, respectively.

Proposition 6.2. *We have:*

$$\varphi_n(e_{i_1} \cdots e_{i_n}) = \sum_{\tau \in S_n} \text{sg}(\tau) \otimes x_{i_{\tau(1)}} \otimes \cdots \otimes x_{i_{\tau(n)}} \otimes 1$$

and

$$\phi_n(1 \otimes x_{i_1} \otimes \cdots \otimes x_{i_n} \otimes 1) = \frac{1}{n!} e_{i_1} \cdots e_{i_n}.$$

Proof. The first equality it follows easily by induction on n . We now prove the second one also by induction on n . The cases $n = 0$ and $n = 1$ are direct and very simple. Assume that $n > 1$. By definition

$$\phi(1 \otimes x_{i_1} \otimes \cdots \otimes x_{i_n} \otimes 1) = \sigma \circ \phi \circ b'(1 \otimes x_{i_1} \otimes \cdots \otimes x_{i_n} \otimes 1).$$

Since σ is left H -linear and $\sigma \circ \sigma = 0$,

$$\sigma \circ \phi(b'(1 \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}) \otimes 1) = \sigma \circ \sigma \circ \phi \circ b'(b'(1 \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}) \otimes 1) = 0.$$

So,

$$\begin{aligned}
\phi(1 \otimes x_{i_1} \otimes \cdots \otimes x_{i_n} \otimes 1) &= (-1)^n \sigma \circ \phi(1 \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}) \\
&= \frac{(-1)^n}{(n-1)!} \sigma(e_{i_1} \cdots e_{i_{n-1}} Z_{i_n}) \\
&= \frac{(-1)^n}{(n-1)!} \sigma(e_{i_1} \cdots e_{i_{n-1}} (Z_{i_n} - Y_{i_n})) \\
&\quad + \frac{(-1)^n}{(n-1)!} \sigma(e_{i_1} \cdots e_{i_{n-1}} Y_{i_n}) \\
&= \frac{1}{n!} e_{i_1} \cdots e_{i_n}.
\end{aligned}$$

as we want. \square

7. BRAIDED CROSSED PRODUCTS OF $k[X_1, X_2]$

Let $H = k[X_1, X_2]$ be the polynomial k -algebra in two variables endowed with the usual Hopf algebra structure and let (A, s) be a left H -module algebra. Let $\alpha_i^j: A \rightarrow A$ for $1 \leq i, j \leq 2$, such that $s(X_i \otimes a) = \alpha_i^1(a) \otimes X_1 + \alpha_i^2(a) \otimes X_2$ and let $\overline{\alpha}(a)$ be as in Section 6.

Let $\rho: H \otimes A \rightarrow A$ denote the action of H on (A, s) . We write $\rho(h \otimes a) = h \cdot a$ and $X_i \cdot a = \beta_i(a)$. Set

$$\overline{\beta}(a) = \begin{pmatrix} \beta_1(a) \\ \beta_2(a) \end{pmatrix}.$$

Note that the conditions

$$\begin{aligned}
\rho \circ (H \otimes \mu) &= \mu \circ (\rho \otimes \rho) \circ (H \otimes s \otimes A) \circ (\Delta \otimes A \otimes A), \\
h \cdot 1 &= \epsilon(h)1 \quad \text{for all } h \in H, \\
h \cdot (l \cdot a) &= (hl) \cdot a \quad \text{for all } h, l \in H \text{ and } a \in A, \\
s \circ (H \otimes \rho) &= (\rho \otimes H) \circ (H \otimes s) \circ (c \otimes A),
\end{aligned}$$

are equivalent to

$$(3) \quad \overline{\beta}(ab) = \overline{\beta}(a)b + \overline{\alpha}(a)\overline{\beta}(b),$$

$$(4) \quad \beta_2 \circ \beta_1 = \beta_1 \circ \beta_2,$$

$$(5) \quad \beta_l \circ \alpha_i^j = \alpha_i^j \circ \beta_l, \quad \text{for all } i, j, l \in \{1, 2\}.$$

7.1. A simple resolution. For the Hopf algebra $H = k[X_1, X_2]$, the relative resolution of (H, s) constructed in Section 6 becomes

$$(6) \quad (H, s) \xleftarrow{\mu} (D_0, s_{D_0}) \xleftarrow{d_1} (D_1, s_{D_1}) \xleftarrow{d_2} (D_2, s_{D_2}) \xleftarrow{\quad} 0,$$

where (D_*, d_*) is the differential graded k -algebra generated by variables Y_1, Y_2, Z_1, Z_2 in degree 0 and e_1, e_2 in degree 1, subject to the following relations

- (1) Y_1, Y_2, Z_1, Z_2 commute between them,
- (2) $e_i Y_j = Y_j e_i$ and $e_i Z_j = Z_j e_i$ for $i, j \in \{1, 2\}$,
- (3) $e_1^2 = e_2^2 = 0$ and $e_1 e_2 = -e_2 e_1$,

with differential d_* defined by $d_*(e_i) = Y_i - Z_i$ for $i \in \{1, 2\}$, and μ is the H -bimodule map given by $\mu(1) = 1$. In Section 6 it was given explicit formulas for and a contracting homotopy σ_0, σ_1 and σ_2 of (6).

7.2. Computing the cohomology. Applying the functor Ξ to $((D_*, s_{D_*}), d_*)$ we obtain the cochain complex

$$(7) \quad 0 \longrightarrow \Xi(D_0) \xrightarrow{d^1} \Xi(D_1) \xrightarrow{d^2} \Xi(D_2) \longrightarrow 0.$$

Let U be a set of generators of A as a k -algebra. From Propositions 5.5 and 5.6 it follows immediately that

$$\begin{aligned} \Xi(D_0) &= \{f \in \text{Hom}_{H^e}(D_0, Z(^sA)^+) : f(1) \in Z(A)\} \simeq {}^sA \cap Z(A), \\ \Xi(D_1) &= \{f \in \text{Hom}_{H^e}(D_1, Z(^sA)^+) : f(e_i)a = \alpha_i^1(a)f(e_1) + \alpha_i^2(a)f(e_2) \ \forall a \in U\} \\ &\simeq \{(b_1, b_2) \in Z(^sA) \times Z(^sA) : b_i a = \alpha_i^1(a)b_1 + \alpha_i^2(a)b_2 \ \forall a \in U\}, \\ \Xi(D_2) &= \{f \in \text{Hom}_{H^e}(D_2, Z(^sA)^+) : f(e_1 e_2)a = (\alpha_1^1 \circ \alpha_2^2 - \alpha_1^2 \circ \alpha_2^1)(a)f(e_1 e_2) \ \forall a \in U\} \\ &\simeq \{b \in Z(^sA) : ba = (\alpha_1^1 \circ \alpha_2^2 - \alpha_1^2 \circ \alpha_2^1)(a)b \ \forall a \in U\}. \end{aligned}$$

The boundary maps are given by

$$d^1(b) = (\beta_1(b), \beta_2(b)) \quad \text{and} \quad d^2(b_1, b_2) = \beta_1(b_2) - \beta_2(b_1).$$

Let (C_s^*, δ^*) be the complex introduced at the beginning of Section 5. The map ϕ_* induces a quasi-isomorphism $\phi^* : (\Xi(D_*), d^*) \rightarrow (C_s^*, \delta^*)$. By Proposition 6.2,

$$\phi^2(b)(X_1 \otimes X_2) = -\phi^2(b)(X_2 \otimes X_1) = \frac{1}{2}b.$$

Let us write $f = \exp(\phi^2(b))$. It is easy to check that

$$(8) \quad f(X_1 \otimes X_2) = -f(X_2 \otimes X_1) = \frac{1}{2}b.$$

From the formula (7) in [G-G1, Section 10] it follows easily that $A \#_f H$ is generated the elements $a \in A$ and $W_i = 1 \# X_i$ with $i = 1, 2$. Using this, the formulas for s and ρ obtained at the beginning of this section and equality (8), it is easy to see that $A \#_f H$ is isomorphic to the algebra with underlying left A -module structure $A[W_1, W_2]$ and multiplication given by

$$W_i a = \alpha_i^1(a)W_1 + \alpha_i^2(a)W_2 + \beta_i(a) \quad \text{and} \quad W_1 W_2 - W_2 W_1 = b,$$

where i runs on $\{1, 2\}$ and $a \in A$.

7.3. A concrete example. In this subsection given a matrix B we let B_{ij} denote its (i, j) entry. Assume that $A = k[Y]$ and that there exists a matrix $Q \in \text{GL}_2(k)$, such that

$$\overline{\alpha}(Y) = QY = \begin{pmatrix} Q_{11}Y & Q_{12}Y \\ Q_{21}Y & Q_{22}Y \end{pmatrix}.$$

Since $\overline{\alpha}(ab) = \overline{\alpha}(a)\overline{\alpha}(b)$ for all $a, b \in A$, this implies that $\overline{\alpha}(Y^n) = Q^n Y^n$. Hence,

$$\alpha_i^j(Y^n) = (Q^n)_{ij} Y^n = \sum_{j_1, \dots, j_n} Q_{i,j_1} Q_{j_1,j_2} \dots Q_{j_{n-1},j_n} Q_{j_n,j} Y^n \quad \text{for all } i, j.$$

We are going to characterize the maps $\beta_l : k[Y] \rightarrow k[Y]$ ($l = 1, 2$), satisfying (3), (4) and (5). An inductive argument shows that condition (3) hold if and only if $\beta(1) = 0$ and

$$(9) \quad \overline{\beta}(Y^n) = (\text{id} + Q + \dots + Q^{n-1})Y^{n-1}\overline{\beta}(Y) \quad \text{for } n \geq 1.$$

Write $\beta_l(Y) = \sum_{u=0}^{\infty} b_u^{(l)} Y^u$ and $Q^{(n)} = \text{id} + Q + \dots + Q^{n-1}$ (of course $b_u^{(l)} = 0$ except for a finite number of terms). Next, we are going to determine necessary

and sufficient conditions in order that also $\beta_l \circ \alpha_i^j = \alpha_i^j \circ \beta_l$ for all $i, j, l \in \{1, 2\}$. It is immediate that $\beta_l \circ \alpha_i^j(1) = \alpha_i^j \circ \beta_l(1)$. Furthermore, by equation (9), we have

$$\begin{aligned}\beta_l(\alpha_i^j(Y^n)) &= (Q^n)_{ij} \beta_l(Y^n) \\ &= (Q^n)_{ij} (Q_{l1}^{(n)} Y^{n-1} \beta_1(Y) + Q_{l2}^{(n)} Y^{n-1} \beta_2(Y)) \\ &= (Q^n)_{ij} \sum_{u=-1}^{\infty} (Q_{l1}^{(n)} b_{u+1}^{(1)} + Q_{l2}^{(n)} b_{u+1}^{(2)}) Y^{n+u}\end{aligned}$$

and

$$\begin{aligned}\alpha_i^j(\beta_l(Y^n)) &= \alpha_i^j(Q_{l1}^{(n)} Y^{n-1} \beta_1(Y) + Q_{l2}^{(n)} Y^{n-1} \beta_2(Y)) \\ &= \sum_{u=-1}^{\infty} \alpha_i^j((Q_{l1}^{(n)} b_{u+1}^{(1)} + Q_{l2}^{(n)} b_{u+1}^{(2)}) Y^{n+u}) \\ &= \sum_{u=-1}^{\infty} (Q_{l1}^{(n)} b_{u+1}^{(1)} + Q_{l2}^{(n)} b_{u+1}^{(2)}) (Q^{n+u})_{ij} Y^{n+u},\end{aligned}$$

for $n \geq 1$. Hence, $\beta_l \circ \alpha_i^j = \alpha_i^j \circ \beta_l$ for all $i, j, l \in \{1, 2\}$ if and only if

$$(10) \quad (Q_{l1}^{(n)} b_{u+1}^{(1)} + Q_{l2}^{(n)} b_{u+1}^{(2)})(Q^u - \text{id}) = 0 \quad \text{for all } n \geq 1, l \in \{1, 2\} \text{ and } u \geq -1.$$

Note that the case $n = 1$ is clearly equivalent to

$$(11) \quad Q^u = \text{id} \quad \text{or} \quad b_{u+1}^{(1)} = b_{u+1}^{(2)} = 0 \quad \text{for all } u \geq -1.$$

Conversely, it is immediate that from these equalities it follows (10). It remains to determine necessary and sufficient conditions in order that also $\beta_1 \circ \beta_2 = \beta_2 \circ \beta_1$. Since $\beta_1 \circ \beta_2(1) = \beta_2 \circ \beta_1(1)$ we only need to compare $\beta_1 \circ \beta_2(Y^n)$ with $\beta_2 \circ \beta_1(Y^n)$ for $n \geq 1$. We consider several cases.

1) $Q^n \neq \text{id}$ for all $n \in \mathbb{N}$: In this case condition (11) implies that there exist $b^{(1)}, b^{(2)} \in k$ such that $\beta_1(Y) = b^{(1)}Y$ and $\beta_2(Y) = b^{(2)}Y$, and so by equation (9)

$$\beta_2 \circ \beta_1(Y^n) = (Q_{21}^{(n)} b^{(1)} + Q_{22}^{(n)} b^{(2)})(Q_{11}^{(n)} b^{(1)} + Q_{12}^{(n)} b^{(2)}) Y^n = \beta_1 \circ \beta_2(Y^n),$$

for all $n \geq 1$.

2) $Q = \text{id}$: In this case $Q^{(n)} = n \text{id}$, and so by equation (9)

$$\beta_2 \circ \beta_1(Y^n) = \sum_{u=0}^{\infty} n b_u^{(1)} \beta_2(Y^{n+u-1}) = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} n(n+u-1) b_u^{(1)} b_v^{(2)} Y^{n+u+v-2}$$

and

$$\beta_1 \circ \beta_2(Y^n) = \sum_{v=0}^{\infty} n b_v^{(2)} \beta_1(Y^{n+v-1}) = \sum_{v=0}^{\infty} \sum_{u=0}^{\infty} n(n+v-1) b_u^{(1)} b_v^{(2)} Y^{n+u+v-2}.$$

Hence $\beta_2 \circ \beta_1 = \beta_1 \circ \beta_2$ if and only if

$$(12) \quad \sum_{u+v=r} (u-v) b_u^{(1)} b_v^{(2)} = 0 \quad \text{for all } r \geq 0.$$

These conditions are obviously satisfied if $\beta_1 = 0$ or $\beta_2 = 0$. Assume that $\beta_1 \neq 0$ and $\beta_2 \neq 0$. Let $i_1 = \min\{i : b_i^{(1)} \neq 0\}$ and $i_2 = \min\{i : b_i^{(2)} \neq 0\}$. From (12) it follows that $(i_1 - i_2) b_{i_1}^{(1)} b_{i_2}^{(2)} = 0$, which implies that $i_1 = i_2$. Write $c = b_{i_1}^{(2)} / b_{i_1}^{(1)}$. We

claim that $b_i^{(2)} = cb_i^{(1)}$ for all $i \geq i_1$. Suppose this fact is true for $i \in \{i_1, \dots, j-1\}$. Then, again by (12), we have

$$\begin{aligned} 0 &= \sum_{u+v=i_1+j} (u-v)b_u^{(1)}b_v^{(2)} \\ &= (i_1-j)b_{i_1}^{(1)}b_j^{(2)} + c \sum_{\substack{u+v=i_1+j \\ v < j}} (u-v)b_u^{(1)}b_v^{(1)} \\ &= (i_1-j)b_{i_1}^{(1)}b_j^{(2)} + c(j-i_1)b_j^{(1)}b_{i_1}^{(1)}, \end{aligned}$$

which implies that $b_j^{(2)} = cb_j^{(1)}$. Conversely, it is easy to check that if $b_i^{(2)} = cb_i^{(1)}$ for all $i \geq i_1$, then the equation (12) is satisfied.

3) $Q \neq \text{id}$ has finite order: Let $m > 1$ be the order of Q . By condition (11), we know that $\beta_1(Y), \beta_2(Y) \in Yk[Y^m]$. Since $X^m - 1$ has simple roots, Q is diagonalizable. By mean of a linear change of variables in $k[X_1, X_2]$ we can replace Q by a diagonal matrix whose entries are m -th roots of unity (so we are replacing X_1 and X_2 for appropriate linear combinations of them). For simplicity we also call X_1 and X_2 the new variables and we keep the name Q for the new matrix associated with s and β_1, β_2 for the new k -linear endomorphisms of $k[Y]$ defining the action of H on $(k[Y], s)$. Since Q is diagonal the matrices $Q^{(n)}$'s are also, and if $n = mq_n + r_n$, then $Q^{(n)} = q_n Q^{(m)} + Q^{(r_n)}$. Hence, by equation (9),

$$\begin{aligned} \beta_2 \circ \beta_1(Y^n) &= \sum_{u=0}^{\infty} Q_{11}^{(n)} b_{mu+1}^{(1)} \beta_2(Y^{n+mu}) \\ &= \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} Q_{11}^{(n)} Q_{22}^{(n+mu)} b_{mu+1}^{(1)} b_{mv+1}^{(2)} Y^{n+mu+mv} \\ &= \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} q_n (q_n+u) Q_{11}^{(m)} Q_{22}^{(m)} b_{mv+1}^{(2)} b_{mu+1}^{(1)} Y^{n+mu+mv} \\ &\quad + \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} (q_n+u) Q_{11}^{(r_n)} Q_{22}^{(m)} b_{mu+1}^{(1)} b_{mv+1}^{(2)} Y^{n+mu+mv} \\ &\quad + \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} q_n Q_{11}^{(m)} Q_{22}^{(r_n)} b_{mu+1}^{(1)} b_{mv+1}^{(2)} Y^{n+mu+mv} \\ &\quad + \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} Q_{11}^{(r_n)} Q_{22}^{(r_n)} b_{mu+1}^{(1)} b_{mv+1}^{(2)} Y^{n+mu+mv}, \end{aligned}$$

and similarly

$$\begin{aligned} \beta_1 \circ \beta_2(Y^n) &= \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} q_n (q_n+v) Q_{11}^{(m)} Q_{22}^{(m)} b_{mu+1}^{(1)} b_{mv+1}^{(2)} Y^{n+mu+mv} \\ &\quad + \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} (q_n+v) Q_{11}^{(m)} Q_{22}^{(r_n)} b_{mu+1}^{(1)} b_{mv+1}^{(2)} Y^{n+mu+mv} \\ &\quad + \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} q_n Q_{11}^{(r_n)} Q_{22}^{(m)} b_{mu+1}^{(1)} b_{mv+1}^{(2)} Y^{n+mu+mv} \\ &\quad + \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} Q_{11}^{(r_n)} Q_{22}^{(r_n)} b_{mu+1}^{(1)} b_{mv+1}^{(2)} Y^{n+mu+mv}. \end{aligned}$$

So $\beta_2 \circ \beta_1(Y^n) = \beta_1 \circ \beta_2(Y^n)$ if and only if

$$\begin{aligned} 0 &= \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} q_n(u-v) Q_{11}^{(m)} Q_{22}^{(m)} b_{mv+1}^{(2)} b_{mu+1}^{(1)} Y^{n+mu+mv} \\ &\quad + \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} u Q_{11}^{(r_n)} Q_{22}^{(m)} b_{mu+1}^{(1)} b_{mv+1}^{(2)} Y^{n+mu+mv} \\ &\quad - \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} v Q_{11}^{(m)} Q_{22}^{(r_n)} b_{mu+1}^{(1)} b_{mv+1}^{(2)} Y^{n+mu+mv}. \end{aligned}$$

If Q_{11} and Q_{22} are both different than 1, then $Q_{11}^{(m)} = Q_{22}^{(m)} = 0$ and the above expression vanishes. It remains to consider the case $Q_{11} \neq Q_{22}$ and $1 \in \{Q_{11}, Q_{22}\}$. Without loose of generality we can consider that $Q_{22} = 1$. In this case the above equality becomes

$$(13) \quad 0 = \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} um Q_{11}^{(r_n)} b_{mu+1}^{(1)} b_{mv+1}^{(2)} Y^{n+mu+mv}.$$

From all these facts it follows that it must be

$$\beta_1 \in Yk[Y^m] \text{ and } \beta_2 = 0, \quad \text{or } \beta_1(Y) = b^{(1)}Y \text{ and } \beta_2 \in Yk[Y^m] \setminus \{0\}.$$

7.3.1. Classification of the crossed products $k[Y]\#k[X_1, X_2]$. By the general theory of braided crossed products, we know that the underlying vector space of $k[Y]\#k[X_1, X_2]$ is $k[Y, W_1, W_2]$, where $W_i = 1\#X_i$, and Y, W_1 and W_2 generate $k[Y]\#k[X_1, X_2]$ as a k -algebra. Next, we classify these crossed products in each of the cases considered above. To carry out this task we use the complex (7). We assume that Q is a Jordan Matrix (if this is not the case, but k is algebraically closed, then we can replace X_1 and X_2 by convenient linear combinations of them, in such a way that the matrix associated with s let be a Jordan Matrix).

1) $Q^n \neq \text{id}$ for all $n \in \mathbb{N}$: We know that there exist $b^{(1)}$ and $b^{(2)}$ in k such that $\beta_1(Y) = b^{(1)}Y$ and $\beta_2(Y) = b^{(2)}Y$. There are two possibilities:

$$1a. \quad Q = \begin{pmatrix} q & 1 \\ 0 & q \end{pmatrix} \text{ with } q \neq 0,$$

$$1b. \quad Q = \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \text{ with } q_1, q_2 \in k \setminus \{0\} \text{ and } q_1 \text{ or } q_2 \text{ a non root of 1.}$$

We consider first the case 1a. An easy computation shows that $A^s = k$. From this it follows immediately that $d^2 = 0$ and

$$\Xi(D_2) = \begin{cases} 0 & \text{if } q^2 \neq 1, \\ k & \text{if } q^2 = 1. \end{cases}$$

Hence; if $q^2 \neq 1$, then the multiplicative structure of $k[Y]\#k[X_1, X_2]$ is determined by the relations

$W_1Y = qYW_1 + YW_2 + b^{(1)}Y, \quad W_2Y = qYW_2 + b^{(2)}Y \quad \text{and} \quad W_1W_2 - W_2W_1 = 0,$
with $b^{(1)}, b^{(2)} \in k$; and if $q^2 = 1$, then it is determined by the relations

$W_1Y = qYW_1 + YW_2 + b^{(1)}Y, \quad W_2Y = qYW_2 + b^{(2)}Y \quad \text{and} \quad W_1W_2 - W_2W_1 = \lambda,$
with $b^{(1)}, b^{(2)}, \lambda \in k$.

We consider now the case 1b. An easy computation shows that $A^s = k$. From this it follows immediately that $d^2 = 0$ and

$$\Xi(D_2) = \begin{cases} 0 & \text{if } q_1q_2 \neq 1, \\ k & \text{if } q_1q_2 = 1. \end{cases}$$

Hence; if $q_1 q_2 \neq 1$, then the multiplicative structure of $k[Y] \# k[X_1, X_2]$ is determined by the relations

$$W_1 Y = q_1 Y W_1 + b^{(1)} Y, \quad W_2 Y = q_2 Y W_2 + b^{(2)} Y \quad \text{and} \quad W_1 W_2 - W_2 W_1 = 0,$$

with $b^{(1)}, b^{(2)} \in k$; and if $q_1 q_2 = 1$, then it is determined by the relations

$$W_1 Y = q_1 Y W_1 + b^{(1)} Y, \quad W_2 Y = q_2 Y W_2 + b^{(2)} Y \quad \text{and} \quad W_1 W_2 - W_2 W_1 = \lambda,$$

with $b^{(1)}, b^{(2)}, \lambda \in k$.

2) $Q = \text{id}$ (The classical case): By the discussion above we know that $\beta_1 = 0$ or there exists $c \in k$ such that $\beta_2(Y) = c\beta_1(Y)$. An easy computation shows that

$$\Xi(D_1) = k[Y] \times k[Y], \quad \Xi(D_2) = k[Y]$$

and

$$d^2(Y^r, Y^s) = sY^{s-1}\beta_1(Y) - rY^{r-1}\beta_2(Y) = \begin{cases} -rY^{r-1}\beta_2(Y) & \text{if } \beta_1 = 0, \\ (sY^{s-1} - rY^{r-1}c)\beta_1(Y) & \text{if } \beta_1 \neq 0, \end{cases}$$

where for the computing of d^2 we have used (9). So,

$$\mathbb{H}^2(\Xi(D_*), d^*) = \begin{cases} \frac{k[Y]}{\langle \beta_2(Y) \rangle} & \text{if } \beta_1 = 0, \\ \frac{k[Y]}{\langle \beta_1(Y) \rangle} & \text{if } \beta_1 \neq 0. \end{cases}$$

Hence; if $\beta_1 \neq 0$, then the multiplicative structure of $k[Y] \# k[X_1, X_2]$ is determined by the relations

$$W_1 Y = Y W_1 + \beta_1(Y), \quad W_2 Y = Y W_2 + c\beta_1(Y) \quad \text{and} \quad W_1 W_2 - W_2 W_1 = R(Y),$$

where $R(Y) = 0$ or $\text{dg}(R(Y)) < \text{dg}(\beta_1(Y))$; if $\beta_1 = 0$ and $\beta_2 \neq 0$, then it is determined by the relations

$$W_1 Y = Y W_1, \quad W_2 Y = Y W_2 + \beta_2(Y) \quad \text{and} \quad W_1 W_2 - W_2 W_1 = R(Y),$$

where $R(Y) = 0$ or $\text{dg}(R(Y)) < \text{dg}(\beta_2(Y))$; and if $\beta_1 = \beta_2 = 0 \neq 0$, then it is determined by the relations

$$W_1 Y = Y W_1, \quad W_2 Y = Y W_2 \quad \text{and} \quad W_1 W_2 - W_2 W_1 = R(Y),$$

where $R(Y)$ is an arbitrary polynomial.

3) $Q \neq \text{id}$ has finite order $m > 0$: Then Q is a diagonal matrix whose diagonal entries q_1 and q_2 are roots of unity of order m_1 and m_2 and the lowest common multiple of m_1 and m_2 is m . We can reduce to the following two possibilities:

1a. $q_1, q_2 \neq 1$. In this case $\beta_1(Y), \beta_2(Y) \in Yk[Y^m]$.

1b. q_1 a root of unity of order m and $q_2 = 1$. In this case $\beta_1(Y) \in Yk[Y^m]$ and $\beta_2(Y) = 0$ or there exists $b^{(1)} \in k$ such that $\beta_1(Y) = b^{(1)}Y$ and $\beta_2(Y) \in Yk[Y^m] \setminus \{0\}$.

We consider first the case 1a. An easy computation shows that $\Xi(D_1) = 0$ (hence $d^2 = 0$) and

$$\Xi(D_2) = \begin{cases} 0 & \text{if } q_1 q_2 \neq 1, \\ k[Y^m] & \text{if } q_1 q_2 = 1. \end{cases}$$

Hence; if $q_1 q_2 \neq 1$, then the multiplicative structure of $k[Y] \# k[X_1, X_2]$ is determined by the relations

$$W_1 Y = q_1 Y W_1 + \beta_1(Y), \quad W_2 Y = q_2 Y W_2 + \beta_2(Y) \quad \text{and} \quad W_1 W_2 - W_2 W_1 = 0,$$

with $\beta_1(Y)$ and $\beta_2(Y)$ belong to $Yk[Y^m]$; and if $q_1q_2 = 1$, then it is determined by the relations

$$W_1Y = q_1YW_1 + \beta_1(Y), \quad W_2Y = q_2YW_2 + \beta_2(Y) \quad \text{and} \quad W_1W_2 - W_2W_1 = P(Y^m),$$

with $\beta_1(Y), \beta_2(Y) \in Yk[Y^m]$ and $P \in k[Y]$.

We consider now the case 1b. An easy computation shows that $\Xi(D_2) = 0$. Hence, the multiplicative structure of $k[Y] \# k[X_1, X_2]$ is determined by the relations

$$W_1Y = q_1YW_1 + \beta_1(Y), \quad W_2Y = YW_2 + \beta_2(Y) \quad \text{and} \quad W_1W_2 - W_2W_1 = 0,$$

with $\beta_1(Y) \in Yk[Y^m]$ and $\beta_2(Y) = 0$ or $\beta_1(Y) = b^{(1)}Y$ and $\beta_2(Y) \in Yk[Y^m] \setminus \{0\}$.

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